

SOUTHWESTERN PONDEROSA PINE, DOUGLAS-FIR AND
WHITE FIR VOLUME EQUATIONS AND TAPER FUNCTIONS

by

John Paul McTague

Zheng Lan

William F. Stansfield

A Report Submitted to

USDA Forest Service, Southwestern Region and
Rocky Mountain Forest & Range Experiment Station

Stone Forest Industries, Inc.

Kaibab Forest Products Company

Northern Arizona University

School of Forestry

May 1992

The authors are Associate Professor, former Graduate Student,
and Senior Research Specialist, Northern Arizona University,
School of Forestry, Box 4098, Flagstaff, AZ 86011-4098.

The authors gratefully acknowledge the assistance of Edwin J.
Green for explaining the EB methodology used in this study.

LIBRARY COPY
ROCKY MTN. FOREST & RANGE
EXPERIMENT STATION

TABLE OF CONTENTS

List of Tables	v
List of Figures	xiii
Chapter 1: Introduction	1
Chapter 2: Literature Review	6
Review of Total Volume Equations	6
Review of Merchantable Volume Equations and Taper Functions	17
Application of New Techniques in Developing Volume Equations	26
Chapter 3: Methods	30
Sources and Nature of Data	30
Field Measurement	40
Computation of Observed Butt Log Cubic Foot Volume .	41
Computation of Cubic Foot Log Volume	45
Computation of Scribner Board Foot Log Volume . . .	46
Chapter 4: Analysis and Results	48
Empirical Bayes Estimator	52
ETS Estimator	53
Application to Volume Models	55
Total Volume	55
Total Inside Bark Cubic Foot Volume Equation . .	55
Total Outside Bark Cubic Foot Volume Equation . .	73
Total Scribner Board Foot Volume Equation	75

Merchantable Volume and Taper Function	83
Merchantable Inside Bark Cubic	
Foot Volume Equations	83
Merchantable Outside Bark Cubic	
Foot Volume Equation	116
Merchantable Scribner Board	
Foot Volume Equations	118
Inside Bark Taper Function	143
Outside Bark Upper-stem Diameter Equation	162
Chapter 5: Conclusion	165
Evaluation	165
Conclusion	166
Literature Cited	170

LIST OF TABLES

1.	Frequency of Sample Trees by DBH and Height Classes - Ponderosa Pine	33
2.	Distribution of Sample Trees from 12 National Forests - Ponderosa Pine	34
3.	Frequency of Sample Trees by DBH and Height Classes - Douglas-Fir	36
4.	Frequency of Sample Trees by DBH and Height Classes - White Fir	37
5.	Distribution of Sample Trees from 7 National Forests - Douglas-Fir	38
6.	Distribution of Sample Trees from 7 National Forests - White Fir	38
7.	Regression Coefficients, Join Point and Observed D ² H Range of the Segmented Polynomial Total Volume Equations for Ponderosa Pine by National Forest	51
8.	Coefficients for the Total Inside Bark Cubic Foot Volume Equation from OLS Regression by National Forest - Ponderosa Pine	58
9.	Coefficients for the Total Inside Bark Cubic Foot Volume Equation from OLS Regression by National Forest - Douglas-Fir	59
10.	Coefficients for the Total Inside Bark Cubic Foot Volume Equation from OLS Regression by National Forest - White Fir	60
11.	Estimated Hyperparameter Coefficients of Equation (5) and Associated t-test - Ponderosa Pine	63
12.	Estimated Hyperparameter Coefficients of Equation (6) and Associated t-test - Douglas-Fir . .	64

13.	Estimated Hyperparameter Coefficients of Equation (6) and Associated t-test - White Fir . . .	64
14.	Parameters of the EB Estimate for the Total Inside Bark Cubic Foot Volume Equation - Ponderosa Pine	68
15.	Parameters of the EB Estimate for the Total Inside Bark Cubic Foot Volume Equation - Douglas-Fir	69
16.	Parameters of the EB Estimate for the Total Inside Bark Cubic Foot Volume Equation - White Fir	70
17.	The Total Inside Bark Cubic Foot Volume ($V_{t,ib}$) Equations - Ponderosa Pine	71
18.	The Total Inside Bark Cubic Foot Volume ($V_{t,ib}$) Equations - Douglas-Fir	72
19.	The Total Inside Bark Cubic Foot Volume ($V_{t,ib}$) Equations - Douglas-Fir	73
20.	Coefficients for the Total Scribner Board Foot Volume Equation from OLS Regression by National Forest - Ponderosa Pine	78
21.	Coefficients for the Total Scribner Board Foot Volume Equation from OLS Regression by National Forest - Douglas-Fir	79
22.	Coefficients for the Total Scribner Board Foot Volume Equation from OLS Regression by National Forest - White Fir	80
23.	Estimated Hyperparameter Coefficients of Equation (17) and Associated t-test - Ponderosa Pine	82
24.	Estimated Hyperparameter Coefficients of Equation (17) and Associated t-test - Douglas-Fir	82

25.	Estimated Hyperparameter Coefficients of Equation (17) and Associated t-test - White Fir	83
26.	Parameters of the EB Estimate for the Total Scribner Board Foot Volume Equation - Ponderosa Pine	84
27.	Parameters of the EB Estimate for the Total Scribner Board Foot Volume Equation - Douglas-Fir	85
28.	Parameters of the EB Estimate for the Total Scribner Board Foot Volume Equation - White Fir	86
29.	Total Scribner Board Foot Volume ($V_{t,bf}$) Equations - Ponderosa Pine	87
30.	Total Scribner Board Foot Volume ($V_{t,bf}$) Equations - Douglas-Fir	88
31.	Total Scribner Board Foot Volume ($V_{t,bf}$) Equations - White Fir	89
32.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any upper- diameter) from OLS Regression by National Forest - Ponderosa Pine	92
33.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any merchantable height) from OLS Regression by National Forest - Ponderosa Pine	93
34.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any upper- diameter) from OLS Regression by National Forest - Douglas-Fir	94
35.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any upper- diameter) from OLS Regression by National Forest - White Fir	95

36.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any merchantable height) from OLS Regression by National Forest - Douglas-Fir	96
37.	Coefficients for the Merchantable Inside Bark Cubic Foot Volume Equation (for any merchantable height) from OLS Regression by National Forest - White Fir	97
38.	Estimated Hyperparameter Coefficients Based on the First-stage Model (20) and Associated t-test - Ponderosa Pine	99
39.	Estimated Hyperparameter Coefficients Based on the First-stage Model (21) and Associated t-test - Ponderosa Pine	100
40.	Estimated Hyperparameter Coefficients Based on the First-stage Model (24) and Associated t-test - Douglas-Fir	100
41.	Estimated Hyperparameter Coefficients Based on the First-stage Model (24) and Associated t-test - White Fir	101
42.	Estimated Hyperparameter Coefficients Based on the First-stage Model (25) and Associated t-test - Douglas-Fir	101
43.	Estimated Hyperparameter Coefficients Based on the First-stage Model (25) and Associated t-test - White Fir	102
44.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Upper-stem Diameter - Ponderosa Pine	103
45.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Upper-stem Diameter - Douglas-Fir	104
46.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Upper-stem Diameter - White Fir	105
47.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Merchantable Height - Ponderosa Pine	106

48.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Merchantable Height - Douglas-Fir	107
49.	Parameters of the EB Estimate for Inside Bark Merchantable Cubic Foot Volume Based on Merchantable Height - White Fir	108
50.	Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - Ponderosa Pine .	109
51.	Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - Douglas-Fir . . .	112
52.	Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - White Fir	114
53.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any upper-stem diameter) from OLS Estimation by National Forest - Ponderosa Pine	121
54.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any merchantable height) from OLS Estimation by National Forest - Ponderosa Pine	122
55.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any upper-stem diameter) from OLS Estimation by National Forest - Douglas-Fir	123
56.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any upper-stem diameter) from OLS Estimation by National Forest - White Fir	124
57.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any merchantable height) from OLS Estimation by National Forest - Douglas-Fir	125
58.	Coefficients for the Merchantable Scribner Board Foot Volume Equation (for any merchantable height) from OLS Estimation by National Forest - White Fir	126

59.	Estimated Hyperparameter Coefficients Based on the First-stage Model (33) and Associated t-test - Ponderosa Pine	127
60.	Estimated Hyperparameter Coefficients Based on the First-stage Model (34) and Associated t-test - Ponderosa Pine	128
61.	Estimated Hyperparameter Coefficients Based on the First-stage Model (37) and Associated t-test - Douglas-Fir	128
62.	Estimated Hyperparameter Coefficients Based on the First-stage Model (37) and Associated t-test - White Fir	129
63.	Estimated Hyperparameter Coefficients Based on the First-stage Model (38) and Associated t-test - Douglas-Fir	129
64.	Estimated Hyperparameter Coefficients Based on the First-stage Model (38) and Associated t-test - White Fir	130
65.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Upper-stem Diameter - Ponderosa Pine	131
66.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Upper-stem Diameter - Douglas-Fir	132
67.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Upper-stem Diameter - White Fir	133
68.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Merchantable Height - Ponderosa Pine	134
69.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Merchantable Height - Douglas-Fir	135
70.	Parameters of the EB Estimate for Merchantable Scribner Board Foot Volume Based on Merchantable Height - White Fir	136
71.	Merchantable Scribner Board Foot Volume ($V_{m.bf}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - Ponderosa Pine .	137

72.	Merchantable Scribner Board Foot Volume ($V_{m.bf}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - Douglas-Fir . . .	140
73.	Merchantable Scribner Board Foot Volume ($V_{m.bf}$) Equations Based on Upper-stem Diameter (1) and on Merchantable Height (2) - White Fir . . .	142
74.	Estimated Coefficients of the Variable-form Taper Function from OLS Regression by National Forest - Ponderosa Pine . . .	147
75.	Estimated Coefficients of the Variable-form Taper Function from OLS Regression by National Forest - Douglas-Fir . . .	148
76.	Estimated Coefficients of the Variable-form Taper Function from OLS Regression by National Forest - White Fir . . .	149
77.	Estimated Hyperparameter Coefficients of Equation (44), and Associated t-test - Ponderosa Pine . . .	151
78.	Estimated Hyperparameter Coefficients of Equation (45), and Associated t-test - Douglas-Fir . . .	152
79.	Estimated Hyperparameter Coefficients of Equation (44), and Associated t-test - White Fir . . .	153
80.	Parameters of the EB Estimate for the Variable- form Taper Function by National Forest - Ponderosa Pine . . .	154
81.	Parameters of the EB Estimate for the Variable- form Taper Function by National Forest - Douglas-Fir . . .	155
82.	Parameters of the EB Estimate for the Variable- form Taper Function by National Forest - White Fir . . .	156
83.	The Variable-form Inside Bark Taper Function in Non-linear Form by National Forest - Ponderosa Pine . . .	157
84.	The Variable-form Inside Bark Taper Function in Non-linear Form by National Forest - Douglas-Fir . . .	159

85.	The Variable-form Inside Bark Taper Function in Non-linear Form by National Forest - White Fir	161
86.	Bias from an Independent Test of the Total Inside Bark Cubic Foot Volume Equation from OLS for all National Forests Combined, OLS for Each Individual Forest, and EB for Each National Forest	167
87.	SSE from an Independent Test of the Total Inside Bark Cubic Foot Volume Equation from OLS for all National Forests Combined, OLS for each individual Forest, and EB for Each National Forest	168
88.	Overall Average Bias and SSE by Estimation Procedure	169

LIST OF FIGURES

1. Location of the Twelve Southwestern National
Forests Reported in This Study 31

CHAPTER 1

INTRODUCTION

Large portions of the Southwest are covered with commercial timber forests of ponderosa pine (Pinus ponderosa var. scopulorum Engelm.), Douglas-fir (Pseudotsuga menziesii var. glauca [Beissn.] Franco), and white fir (Abies concolor [Gord. and Glend.]). These forests contain vital roundwood needed for the maintenance and future expansion of the forest products industry in the region. During recent history however, some inevitable changes have occurred in the stand structure of many of these forests that merit closer study. As the national forests in the Southwest have come under more intensive management, many of the older and very large diameter stems have been cut. In the future, the target diameter of crop trees at rotation age will be smaller than some of the trees currently being extracted from the forests. There is little doubt that the forest products industry will successfully respond to the challenge of obtaining a greater portion of their procurement needs from smaller trees with more taper. This will be partially accomplished by the incorporation of new technology that improves utilization; however, increasing reliance will be placed upon deriving multiple products from a tree.

With changing merchantability standards in the Southwest, it has become necessary to predict volume to various upper-stem diameters or merchantable heights. For the purpose of merchandising trees into multiple products, taper functions have become essential. Until recently, none of the published merchantable volume equations enabled predicting volume to any desired upper-stem diameter or desired merchantable height for southwestern ponderosa pine. This limitation to the prediction of merchantable volume is still true for Douglas-fir and white fir. Myers (1963) presented ponderosa pine merchantable Scribner volume tables to a minimum top diameter that varied according to the dbh of the tree. The Myers equations however, do not allow the user to select which variable upper-stem diameter to use. The Myers taper tables for southwestern ponderosa pine trees predict scaling diameters of top logs or half-logs. Actual taper equations or statistics of fit were not presented by Myers. The Hann and Bare (1978) equations for ponderosa pine, Douglas-fir, and white fir predict volume to specified upper-stem diameters between only 3 and 8 inches inside bark. McTague and Stansfield (1988) recently constructed ponderosa pine merchantable volume ratio equations and a new taper function that permit the calculation of volume to any upper-stem diameter or merchantable height.

Several volume equations for southwestern ponderosa pine provide distinctly different estimates of volume

depending upon whether the tree is a young "blackjack" tree or an older "yellow pine" (Hornibrook, 1936; Myers, 1963; Ffolliott et al., 1971; Hann and Bare, 1978). For a ponderosa pine tree with bark in a physiological transition between rough black bark and yellow scaly plates, the choice between a blackjack or yellow pine volume equation can yield widely different estimates of tree volume. Other southwestern ponderosa pine volume equations involve the utilization of a system of equations based on whether the combined variable D^2H (the product of dbh squared and total height) is greater or less than some threshold value (USDA Forest Service, 1984; Myers, 1963). The problem associated with this system is that the equations are discontinuous and disjoint when the value of D^2H equals the threshold value. McTague and Stansfield (1988) developed a single continuous volume equation for ponderosa pine that eliminates the need to distinguish between blackjack and yellow pine. Similar to their approach, the development of new ponderosa pine volume equations and taper functions that no longer oblige the user to distinguish between blackjack and yellow pine are presented in this study.

As reliance on the utilization of a wide range of tree sizes increases, it is highly desirable to construct new total volume equations for commercial southwestern species with data sets that contain a wide range of tree sizes and that are from a larger geographic area. The Hann and Bare

(1978) data sets of ponderosa pine were collected from five national forests in Arizona and New Mexico, and their data sets of white fir were obtained from only three national forests in New Mexico. While the majority of the data used in the Hann and Bare study for Douglas-fir were from New Mexico, the only national forest from Arizona was the Tonto N.F., which is predominantly located below the Mogollon Rim. Unfortunately the Hann and Bare data sets have been lost, and hence the bulk of the trees needed to construct new equations and taper functions must come from new sources that hopefully include more national forests in the Southwest. The results of this study are based on the collection of new data sources or old unused data sets from a wide range of geographic locations and growing conditions, and therefore have a better diversity and representation over the Hann and Bare data sets. By employing the Empirical Bayes technique, new volume equations for ponderosa pine, Douglas-fir, and white fir are constructed in this study for individual national forests while simultaneously using parameter averages from all the national forests combined. The results presented in this study for ponderosa pine, Douglas-fir, and white fir may be summarized as the development of new:

1. Total inside bark volume equations in both cubic foot and board foot volume.
2. Inside bark merchantable volume ratio equations in both cubic foot and board foot volume that are a function of any upper-stem diameter or merchantable height.
3. Inside bark taper functions.
4. Outside bark to inside bark relationships for: total volume, merchantable volume, and upper-stem diameter.
5. Volume estimators for butt logs of variable length.

CHAPTER 2

LITERATURE REVIEW

Review of Total Volume Equations

Many volume equations have been developed for commercial timber species in Arizona and New Mexico. Least squares regression has been commonly applied in the construction of volume equations since 1950. Minor (1961) used weighted least squares regression to develop a volume equation for gross or merchantable cubic foot volume to a 4 inch top for pulpwood size ponderosa pine. Myers (1963) used ordinary least squares regression with segmented data to develop volume equations for (a) total stem cubic foot volume, (b) merchantable cubic foot volume to a 4 inch and variable top limit, and (c) International 1/4 inch and Scribner board foot volumes to a variable top limit. Myers used the term "variable top limit" to indicate that the merchantable top diameter was a fixed function of a tree's diameter at breast height and not to indicate that the top diameter was an independent variable in the volume equation. More recently, Hann and Bare (1978) also applied weighted least squares regression to develop volume equations to predict total cubic and board foot volume to a 6 inch or 8 inch top for ponderosa pine, Douglas-fir, and white fir in Arizona and New Mexico. The Hann and Bare (1978) equations for total volume were constructed as a function of the

combined variable D^2H (the product of dbh squared and total height) and are expressed in the following form:

$$V_t = a_0 + a_1(D^2H)$$

where

V_t = total volume in cubic feet

D = dbh outside bark in inches

H = total height in feet

a_0 , a_1 = regression coefficients.

Myers (1963) and Hann and Bare (1978) divided their ponderosa pine data sets by bark characteristics, and separate equations were developed for blackjack and yellow pine. Blackjack pines are immature ponderosa pine with rough dark bark and relatively large taper, while yellow pines are mature ponderosa pine trees with smooth yellow bark and less taper than blackjacks (Myers, 1963; Hann and Bare, 1978). Depending upon the visual techniques that field crews use to classify blackjack or yellow pine, serious differences in sample estimates of standing volume can occur. In one example with a tree which has a diameter of 20 inches and a height of 80 feet, the yellow pine equation predicts a board foot volume which is 19.5 percent greater than the blackjack equation (USDA Forest Service,

1984). For the purpose of computing and projecting tree volume in a growth and yield simulator, it is essential to predict the conversion from blackjack to yellow pine. The equation that Hann (1980) used to estimate the proportion of trees converting from blackjack to yellow pine over a 20-year period was fitted from data of a single virgin stand of very low site index. This equation obviously underestimates the number of trees that attain the yellow pine vigor class on sites of higher quality.

Hornibrook (1936) originally grouped his ponderosa pine data into three groups: blackjack, intermediate, and yellow pine. He developed logarithmic volume equations in Scribner board feet to a 8 inch inside bark top for each of the three physiological groups. Hornibrook found that there no significant difference between the estimated values of the blackjack and intermediate equations, however the estimated values of yellow pine were significantly different from both blackjack and the intermediates. The Hornibrook test for significant difference among the groups was determined using estimated volumes from fitted equations, and many of the values inserted for the predictor variables fell outside the observed data range.

Hann and Bare (1978) did not divide the Douglas-fir data sets nor the white fir data sets by age or vigor. The combined data set for Douglas-fir contained a total of 189 sample trees while the combined data set for white fir

contained a total of 101 trees. They did however, use weighted least squares procedures to determine coefficients of the combined variable volume equation by species and groups of national forests.

In 1984, the Southwestern Region of the U.S. Forest Service determined that volume equations for ponderosa pine would no longer be classified by the blackjack and yellow pine distinction. The new Southwestern Region Forest Service volume equations involve a system of equations by which the selection of the appropriate equation is based on whether the combined variable, D^2H is greater or less than some threshold value (USDA Forest Service, 1984). For example:

- Cubic foot volume (V_t), top diameter is not specified, volume is assumed to be inside bark

- if D^2H of a tree ≤ 33590.92

$$V_t = -1.7751 + 0.0018897 (D^2H)$$

- if D^2H of a tree > 33590.92

$$V_t = -13.542 + 0.00224 (D^2H)$$

- Board foot volume (V_b), Scribner Decimal C

- if D^2H of a tree ≤ 31629.92

$$V_b = -1.786 + 0.00098814 (D^2H)$$

•if D^2H of a tree > 31629.92

$$V_b = -52.897 + 0.12826(HFLL) + \\ 0.0017678(D^2H) + 879120(D^2H)^{-1}$$

where

HFLL = height to the first live limb in feet
(the limb must contain at least one
green needle).

The approach obviously requires the user to measure HFLL or predict HFLL from attributes of tree age and stand density. Another problem associated with this approach is that the equations are discontinuous and disjoint when the D^2H value equals the threshold value (McTague and Stansfield 1988).

The Dixie National Forest uses a system of combined variable volume equations for young growth and old growth Douglas-fir and white fir. The selection of the appropriate equation for the species of interest is based on whether dbh is greater than or less than 20.5 inches. (USDA Forest Service, 1977):

Douglas-fir

$$V = 0.01003D^2H - 25.332 \quad D \leq 20.5 \text{ inches (young growth)}$$

$$V = 0.01011D^2H - 9.552 \quad D \geq 20.6 \text{ inches (old growth)}$$

White fir

$$V = 0.01293D^2H - 34.127 \quad D \leq 20.5 \text{ inches (young growth)}$$

$$V = 0.01218D^2H + 10.603 \quad D \geq 20.6 \text{ inches (old growth)}$$

where

V = Scribner board foot volume to a 6 inch dib top

D = dbh outside bark in inches

H = total height in feet.

This approach tacitly assumes that trees undergo the transition from young growth to old growth when the dbh equals 20.5 inches, regardless of stand density, site quality, or whether tree is a true fir or not. The Intermountain Region volume equations were apparently derived from an earlier study by Kemp (1957), and the revised equations report board foot volume to a 6 inch upper-stem diameter inside bark. An earlier version of these equations reported volume to a variable upper-stem diameter of: $0.3 \text{ dbh} + 2 \text{ inches}$. Similar to the problems encountered with the total volume equations adopted by the Southwestern Region of the Forest Service, the volume equations are discontinuous and disjoint when the value of dbh equals the threshold value.

However, this problem can be overcome using the splining technique outlined by Gallant and Fuller (1973). This technique provides a possibility to join or spline two or more segments into a smooth and continuous function. The join point can be estimated in the regression procedure as a function of attributes such as tree diameter or height. At the join point, the estimate of volume is identical for both

segments. A continuous function is insured by conditioning the first derivative of both segments to be equal at the join point. Chojnacky (1988) has applied the splining technique to develop a single continuous volume equation for juniper, pinyon, oak, and mesquite in Arizona.

McTague and Stansfield (1988) used a logarithmic linear model to develop a single continuous volume equation for ponderosa pine that obviates the need to subjectively distinguish between a blackjack or yellow pine. For their equation, total volume is a function of total tree height, diameter breast height, and Girard form class. Girard form class is defined as the percentage ratio between stem diameter inside bark at the top of the first log (17.3 feet), and the dbh outside bark. By including the Girard form class variable into a model, independent equation parameters for blackjack and yellow pine no longer need to be estimated, and a single volume equation is used for both blackjack and yellow pine. Most of the trees used to construct the McTague and Stansfield equations came from the Eagar lumber recovery study previously reported by Fahey et al. (1986). They were selected in 1981 from 12 areas on National Forests in Arizona and New Mexico. An additional 54 trees were obtained in 1986 by the Southwestern Region from the northern Coconino and southern Kaibab National Forests. The model used by McTague and Stansfield is:

$$V_t = a_0 D^{a_1} H^{a_2} F^{a_3}$$

where

V_t = total volume in cubic feet

D = dbh in inches

H = total height in feet

F = Girard form class

a_i = regression coefficients.

This equation can be linearized using a logarithmic transformation:

$$\ln(V_t) = \ln(a_0) + a_1 \ln(D) + a_2 \ln(H) + a_3 \ln(F)$$

Although this equation has the advantage of providing estimates for both blackjack and yellow pine, it requires the additional measurement of Girard form class. For those occasions where measurements of the Girard form class are not available, these authors have provided a prediction equation for form class that is a function of D^2H .

There are several published merchantable volume equations in Scribner board feet that are in reality, variants of total volume equations. The user of these equations must use either total height or a merchantable height attribute that corresponds to the maximum height of

potential sawlog utilization of the tree. In other words, the merchantable height predictor variable that is inserted into the volume equation is a function of total height, and is not subject to the discretion of the user.

Myers (1963) presented merchantable Scribner volume tables to minimum top diameters that varied according to the dbh of the tree. For instance, on average, the minimum upper-stem diameter for a tree with a dbh of 12.5 is 8 inches, while the minimum upper-stem diameter for a tree with a dbh 41 is 14 inches. The Myers (1963) equations, however, do not let the user decide which variable upper-stem diameter to use. The Myers merchantable cubic foot volume equation to a top limit of sawlog utilization was derived from the model:

$$V_m = a_0 + a_1 (D^2H)$$

Ffolliott et al. (1971) also constructed gross Scribner board foot volume equations for blackjack and old growth southwestern ponderosa pine using the combined variable volume equation. Their height variable, however was based on merchantable tree height in 16-foot logs and half-logs to a variable minimum merchantable diameter. They stated that most blackjack ponderosa pine trees can be utilized to a minimum merchantability limit of 8 inches, while the minimum merchantable diameter in yellow pine timber is more often

governed by top branching characteristics than by diameter. These authors also provided an equation to predict the number of 16-foot logs based on the total tree height of either blackjack or yellow pine trees:

$$N_{16l} = 0.069 H - 1.63$$

where

N_{16l} = number of 16-foot logs

H = total tree height in feet.

The logarithmic linear model has traditionally been popular in the Southwest for expressing merchantable board foot volume of Douglas-fir and white fir to a merchantable height. Unlike the technique used in the Intermountain Region of the Forest Service of constructing a system of equations for each species, no attempt has been made to discriminate volume estimation of Douglas-fir or white fir by either age, vigor, or size class. Lexen and Thompson (1938) developed a white fir merchantable volume table which may be expressed mathematically as:

$$\log(V - 60) = -1.682 + 0.879 \log(D^2 + 10^2) + 1.160 \log(H - 16.3)$$

where

V = Scribner board foot volume to a 10 inch dib top

H = the product of the number of logs and 16.3

D = diameter at breast height inside bark in inches.

log = base 10 logarithms.

Peterson (1958) later developed a similar equation for white fir on the Lincoln National Forest, however to a 8 inch dib merchantability limit. Krauch and Peterson (1943) presented board foot volume tables for Douglas-fir in New Mexico, that were apparently reformulated later with a logarithmic regression model (USDA Forest Service, 1973):

$$\begin{aligned} \log(V - 30) = & -1.8944 + 1.2089 \log(D^2+8^2) \\ & + 0.9238 \log(H - 16.3) \end{aligned}$$

where

V = Scribner board foot volume to a 8 inch dib top
and other variables are defined as above.

These estimated coefficients must contain an error, and the predicted values from the equation above grossly overestimate volume. After refitting the Krauch and Peterson Douglas-fir merchantable volume table to the model above, we estimated the following model coefficients:

$$\begin{aligned} \log(V - 30) = & -1.8968 + 1.0792 \log(D^2+8^2) \\ & + 0.9716 \log(H - 16.3) \end{aligned}$$

This merchantable volume equation of the Krauch and Peterson study for Douglas-fir can be expressed in non-linear form as:

$$V = \frac{1}{78.8497} (D^2 + 64)^{1.0792} (H - 16.3)^{0.9716} + 30$$

Neither Krauch and Peterson (1943) nor Lexen and Thompson (1938) published their diameter breast height outside bark to diameter breast height inside bark relationship, however it may be mathematically inferred from their studies as:

Douglas-fir

$$\text{dbh}_{\text{ob}} = 1.3827 \text{ dbh}_{\text{ib}}^{0.95828}$$

White fir

$$\text{dbh}_{\text{ob}} = 1.3799 \text{ dbh}_{\text{ib}}^{0.94738}$$

Review of Merchantable Volume Equations and Taper Functions

With changing merchantability standards in the Southwest, it has become necessary to predict volume to various upper-stem diameters or merchantable heights. The Hann and Bare (1978) equations for ponderosa pine, Douglas-fir, and white fir predict volume to top diameters between only 3 and 8 inches. The Hann and Bare merchantable volume equation is expressed as follows:

$$\begin{aligned} V_m &= V_t - V_u \\ &= V_t - (b_0 + b_1 d^3 H/D^n + b_2 D^2) \end{aligned}$$

where

V_m = merchantable cubic foot volume which excludes
the volume of the top and stump

V_t = predicted total stem gross cubic foot volume
in an unforked tree

V_u = unmerchantable cubic foot volume

d = inside bark upper-stem diameter in inches

H = total height of the tree in feet

n = a species-national forest specific value of
1.0 or 1.5

b_0 - b_2 = species-national forest specific regression
coefficients.

Burkhart (1977) introduced a merchantable volume ratio equation, based on upper-stem diameter, that eliminated the illogical crossing of predicted volume commonly observed when using independently estimated volume equations for differing upper-stem diameter limits. The Burkhart merchantable volume ratio model was:

$$\frac{V_m}{V_t} = 1 - b_1 \frac{d^{b_2}}{D^{b_3}}$$

Cao and Burkhart (1980) reported the "Modified Burkhart" merchantable volume ratio model, based upon the distance from the tree top to the merchantable height. The "Modified Burkhart" equation was represented as:

$$\frac{V_m}{V_t} = 1 - b_1 \frac{(H-h)^{b_2}}{H^{b_3}}$$

Knoebel et al. (1984) demonstrated a method for indirectly deriving taper and merchantable height functions from the above merchantable volume ratio equations. The Knoebel taper function is noncompatible unless the expression for total volume is derived from Burkhart and Modified Burkhart ratios. Compatible is defined as meaning that the integration of the taper function from the groundline to the total tree height results in an estimate of volume equal to the total volume equation.

McTague and Stansfield (1988) developed merchantable volume ratio equations for ponderosa pine. These equations permit the user to compute cubic foot volume to any upper-stem diameter or any merchantable height. These equations are conditioned so that merchantable volume equals total volume when upper-stem diameter equals zero or when merchantable height equals total height. These equations were fitted with high coefficients of determination. The following inside bark or outside bark merchantable volume ratio equations were estimated from non-linear least squares procedures:

$$V_m = V_t - V_t \frac{a_0 d^{a_1}}{D^{a_2} F^{a_3}} \quad (a)$$

and

$$V_m = V_t - V_t \frac{b_0 (H-h)^{b_1} F^{b_2}}{H^{b_3}} \quad (b)$$

where

h = merchantable height in feet

F = Girard form class

a_i, b_i = regression coefficients

and all other variables are as previously defined.

For the purpose of merchandising trees into multiple products, taper functions and merchantable height equations have become essential. Myers (1963) published taper tables for southwestern ponderosa pine trees that predict scaling top diameters of logs or half-logs. Tabulated scaling diameters represent average tapers for the sample trees measured. Taper tables were constructed by averaging diameters at the taper point. Actual taper equations or statistics of fit were not presented by Myers. McTague and Stansfield (1988) modified an approach used by Knoebel et al. (1984) for indirectly deriving ponderosa pine taper functions and merchantable height functions from

merchantable volume ratio equations. McTague and Stansfield's taper function was derived by equating equations (a) and (b) and is expressed as:

$$d = \left(\frac{b_0 (H-h)^{b_1} F^{b_2+a_3} D^{a_2}}{a_0 H^{b_3}} \right)^{\frac{1}{a_1}}$$

By inverting the taper function, a merchantable height equation was expressed as a function of upper-stem diameter d:

$$h = H - \left(\frac{a_0 d^{a_1} H^{b_3}}{b_0 D^{a_2} F^{b_2+a_3}} \right)^{\frac{1}{b_1}}$$

The taper function presented by McTague and Stansfield (1988) is not compatible. It should be noted that while the McTague and Stansfield taper function performed reasonably well along the entire stem profile without any section of large bias, the function does not minimize the sum of squares of error for upper-stem diameter.

Max and Burkhart (1976) developed a noncompatible taper function which was a flexible segmented splined polynomial model that employs the interaction variable $(d/D)^2$ as the

dependent variable, where d equals upper-stem diameter and D is dbh. This noncompatible taper equation was expressed as:

$$(d/D)^2 = b_1(h/H - 1) + b_2(h^2/H^2 - 1) \\ + b_3(a_1 - h/H)^2 I_1 + b_4(a_2 - h/H)^2 I_2$$

where

$$I_i = 1, \quad \text{if } h/H \leq a_i; \quad i = 1, 2 \\ I_i = 0, \quad \text{if } h/H > a_i.$$

Cao et al. (1980) ranked the performance of various taper functions and merchantable volume ratio equations in their ability to predict upper-stem diameter and merchantable volume. The taper functions were integrated to a merchantable height in order to indirectly estimate volume. Without exception, the merchantable volume ratio model outperformed the taper functions in predicting merchantable volume. Among various merchantable volume ratios, the Burkhart (1977) and "Modified Burkhart" (Cao and Burkhart, 1980) volume ratios gave good estimates and are recommended for predicting merchantable volumes to various heights or top diameters. Among the taper functions, the Max and Burkhart (1976) noncompatible taper function performed the best in predicting upper-stem diameter (Cao et al. 1980). Cao et al. (1980) also found that the simple Ormerod (1973) taper function performed well in predicting

outside bark upper-stem diameter. The simple Ormerod taper function is expressed as:

$$d = D \left(\frac{H-h}{H-4.5} \right)^{b_1}$$

where

b_1 = regression coefficient.

This model is conditioned to predict that upper-stem diameter equals zero at the top of the tree, and that upper-stem diameter equals dbh at breast height. Clutter (1980) later derived a compatible taper function from Burkhart's merchantable volume ratio equation, however its performance was not included in Cao's test (Cao et al. 1980).

Kozak (1988) proposed a variable-form taper model, that predicts the tree profile as a function of total height, dbh, and relative height, with a continuous function using a changing exponent to compensate for the shape of different tree sections. The model has the following form:

$$d = b_0 D^{b_1} b_2^D X^c$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I})$$

$$Z = h/H$$

I = location of the inflection point

$$C = b_3 Z^2 + b_4 \ln(Z+0.001) + b_5 \sqrt{Z} + b_6 e^Z + b_7 (D/H)$$

d = inside bark upper-stem diameter

h = merchantable height

D = outside bark dbh

H = total tree height

b_i = regression coefficients to be estimated.

This model can be linearized using a logarithmic transformation as:

$$\begin{aligned} \ln(d) = & \ln(b_0) + b_1 \ln(D) + \ln(b_2) D + b_3 \ln(X) Z^2 \\ & + b_4 \ln(X) \ln(Z+0.001) + b_5 \ln(X) \sqrt{Z} \\ & + b_6 \ln(X) e^Z + b_7 \ln(X) (D/H) \end{aligned}$$

The linearized taper function can be fitted with ordinary least squares procedures. Among the seven independent variables in Kozak's model, some linear dependencies may exist. When this condition is present, the coefficients from ordinary least squares procedures may not be precisely estimated. Perez et al. (1990) modified the Kozak model and presented a new reduced model by eliminating some of the structural multicollinearities of Kozak's full model. The "best" reduced model was selected based on the following criteria: mean square error (MSE); coefficient of

determination (R^2); and prediction sum of squares (PRESS).

The reduced model is presented as:

$$d = b_0 D^{b_1} X^c$$

where

$$c = b_2 Z^2 + b_3 \ln(Z+0.001) + b_4 (D/H)$$

Linearizing this model using a logarithmic transformation results in:

$$\begin{aligned} \ln(d) = & \ln(b_0) + b_1 \ln(D) + b_2 \ln(X) Z^2 \\ & + b_3 \ln(X) \ln(Z+0.001) + b_4 \ln(X) (D/H) \end{aligned}$$

The variable-form taper models have two weaknesses: numerical integration methods must be used to calculate volume; and iterative methods must be used to find merchantable height to a given diameter. In spite of these shortcomings, the variable-form taper model provides a very useful method for predicting diameter inside bark along the tree stem.

The National Forest System has recently manifested the desire to standardize its procedures for developing taper functions and indirectly estimating the volume of multiple products in the tree. The Forest Service would prefer to

estimate upper-stem diameter at various heights on the tree, and compute volume for each segment with the Smalian formula. Total volume would then be implicitly determined by summing the individual segments. This procedure, advocated by the Forest Service, requires a very precise and unbiased taper function.

Application of New Techniques in Developing Volume Equations

The most widely used volume equations for commercial species in the Southwest are those developed by Hann and Bare (1978). The coefficients of the Hann and Bare equations were estimated for groups of national forests. All the ponderosa pine sample trees used to construct these equations were selected from five national forests; two in Arizona and three in New Mexico. Hann and Bare recommended the use of one equation for the prediction of total volume of blackjack pine on the Coconino, Tonto and Lincoln National Forests and another equation on the Santa Fe and Carson National Forests. One single equation was developed for predicting the total volume of yellow pine on all five of the national forests. The statistical procedure that Hann and Bare used for grouping consisted of analysis of covariance with the logarithmic volume equation and not the combined variable equation. They concluded that blackjack pine, Douglas-fir, and white fir on the Santa Fe and Carson National Forests could not be combined with the same species

on other national forests. There were no sample trees collected on the Kaibab and Apache-Sitgreaves National Forests in Arizona where large acreages of ponderosa pine occur. Hann and Bare did not indicate which equation was appropriate for blackjack pine on the Kaibab and Apache-Sitgreaves National Forests. There are other national forests, such as the Dixie (southern Utah), San Juan and Rio Grande (southwest Colorado), that may be considered to be within the southwestern ponderosa pine type. Hann and Bare (1978) did not include samples from these national forests either.

Some available techniques can be applied to develop volume equations for different national forests simultaneously. Green and Strawderman (1986) applied the Stein-rule method to estimate coefficients for 18 eastern hardwood cubic volume equations. In the Green and Strawderman study, a Stein-rule estimator, which shrink least squares estimates of regression parameters toward their weighted average, was employed to estimate the coefficient in the constant form factor volume equation for 18 species simultaneously. The study results showed that the Stein-rule estimate is biased, although it predicts better than the corresponding least squares estimates, and it also yields lower estimated mean square errors for the volume equation coefficient than the corresponding least squares procedure. Therefore, it is possible to apply the

Stein-rule estimation to take advantage of the similarity expected in individual national forest volume equation coefficients. There is another technique, Empirical Bayes estimation that provides a good estimate of volume equation coefficients. Green and Strawderman (1985) also used Empirical Bayes estimation to develop individual tree volume equations. Essentially the procedure involves combining existing knowledge about the probable value of a parameter with sample evidence to determine an estimate of the parameter or its distribution. Related work on Bayesian estimation (EB) in forest inventory has been reported by Ek and Issos (1978) and Burk and Ek (1982). The Green and Strawderman (1985) study showed that the Empirical Bayes estimators were superior to least squares regression in terms of predictive ability for white oak and black cherry. More recently, Green et al. (1990) applied Empirical Bayes methods for calibrating yield models for Honduran pine to multiple regions. The results showed that the predictive power of the EB estimate was superior to that of multiregion ordinary least squares.

Based on previous work and the present outlook, none of the reported volume equations can entirely meet the changing utilization standards of the forest products industry. The development of new volume equations and taper functions are highly desired for southwestern ponderosa pine, Douglas-fir, and white fir. By including the data of new or unused

sources on more national forests, new volume equations of southwestern commercial species will be constructed using the Empirical Bayes estimation method. The new volume equations will not distinguish the blackjack from the yellow pine, nor do they attempt to distinguish young growth from old growth for Douglas-fir and white fir.

CHAPTER 3

METHODS

Sources and Nature of Data

The ponderosa pine data sets used in this study originate from 12 national forests of the Colorado plateau province and are located in Arizona, New Mexico, southern Utah, and southern Colorado (Figure 1). The largest data set is the Myers (1963) ponderosa pine data set which was collected from the 1930's to the early 1960's. This data set was used to construct the volume and taper tables of southwestern ponderosa pine as reported by Myers (1963). Ponderosa pine data files were retrieved from the Myers (1963) data set for eight specific national forests. They are the Coconino, Kaibab, Tonto, Prescott, and Apache-Sitgreaves National Forests in Arizona, and the Lincoln, Santa Fe, and Carson National Forests in New Mexico. The trees from Myers' Coconino data set were destructively sampled in 1946 from the Fort Valley Experimental Forest. Myers' sample trees from the original Sitgreaves National Forest were collected in 1932 and 1962, while trees from the remaining National Forests of the Myers Study were obtained in 1962. The Myers data set has been retrieved by the U.S. Forest Service, Rocky Mountain Forest and Range Experiment Station and placed in a computer data base. It contains

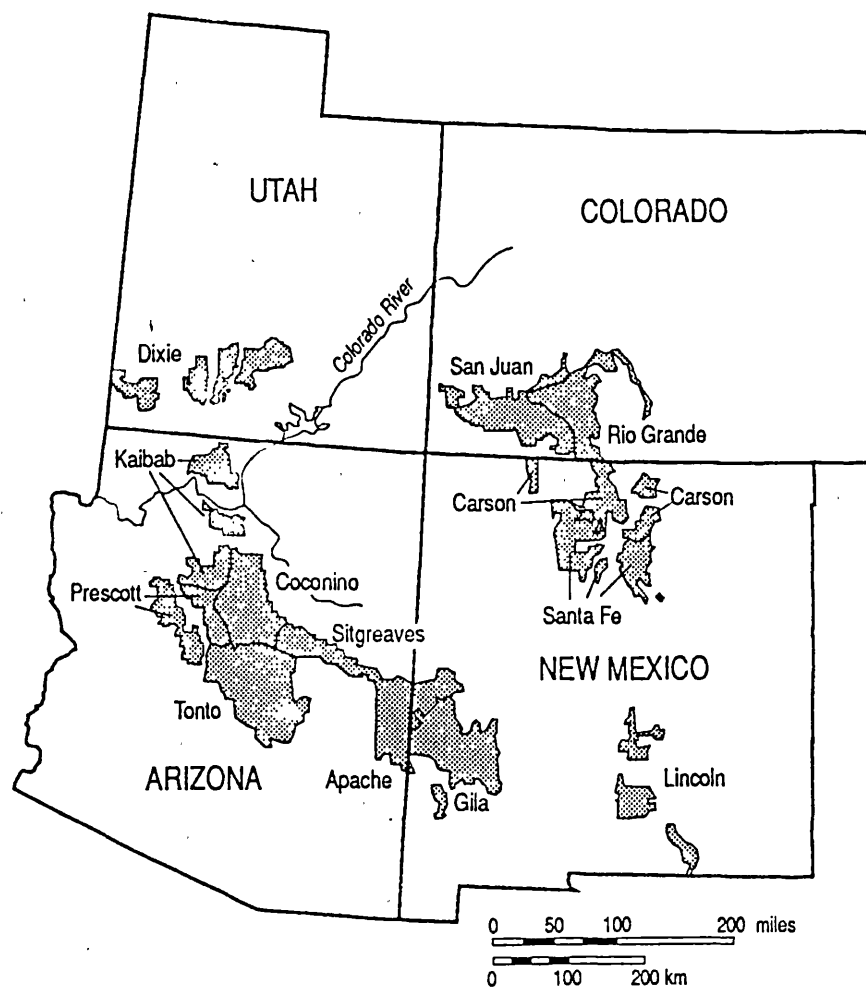


Figure 1. Location of the twelve southwestern national forests reported in this study.

approximately 750 southwestern ponderosa pine trees. Several other data sets, albeit with fewer observations, augment the Myers ponderosa pine data set. Approximately 40 trees were destructively sampled on the Coconino National Forest in 1927 by Hermann Krauch and Bert Lexen and their tree profiles were displayed on graph paper. All of these trees possess a dbh of 5.2 inches or less, and a height of 28.4 feet or less. The diameters and their corresponding heights were read from the graph and placed in a computer data base. This study also contains the data set of the Eagar Mill Lumber Recovery study previously reported by Fahey et al. (1986). The Eagar data set contains tree data information for ponderosa pine from five national forests in Arizona and New Mexico. This data set was collected in 1981 from the Coconino, north district of the Kaibab, Apache-Sitgreaves, and Gila National Forests in Arizona, and the Santa Fe National Forest in New Mexico.

In addition to merging the Myers data set with the Eagar data set, we also obtained data on 22 ponderosa pine trees from the Dixie National Forest in Utah, collected by Kaibab Forest Products Company in 1989. The Rocky Mountain Region of the U.S. Forest Service provided 69 ponderosa pine sample trees from the San Juan National Forest and 11 sample trees from the Rio Grande National Forest. Table 1 indicates the ranges and the frequency of data by dbh classes and total height classes for the ponderosa pine data

Table 1
Frequency of Sample Trees by DBH and Total Height Classes
Ponderosa Pine

dbh (in.)	Total height (ft)											Total
	10.1 -20	20.1 -30	30.1 -40	40.1 -50	50.1 -60	60.1 -70	70.1 -80	80.1 -90	90.1 -100	100.1 -110	110.1 -120	
2.1-4	16											16
4.1-6	6	8										14
6.1-8	4	14	18	21	23	6						86
8.1-10		10	11	28	14	10						73
10.1-12		8	12	25	31	19	10	6				111
12.1-14		5	10	32	40	36	18	8				149
14.1-16			7	30	41	46	23	14	8			169
16.1-18			4	12	38	45	54	22	10			185
18.1-20			2	10	23	27	47	32	17	51		209
20.1-22				8	17	33	54	45	22	10		189
22.1-24				4	10	25	47	51	30	15		182
24.1-26				4	6	16	29	59	31	24		169
26.1-28					8	6	27	34	39	26	1	141
28.1-30						2	12	25	38	25	3	105
30.1-32							8	14	34	19	6	81
Total	26	45	64	174	251	271	329	300	229	170	10	1879

set of this study. Table 2 indicates the distribution of ponderosa pine sample trees from 12 national forests in the Southwest.

Table 2
Distribution of Sample Trees from
12 National Forests

Ponderosa Pine	
National Forest	Number of sample trees
Tonto	64
Apache-Sitgreaves	653
Santa Fe	166
Prescott	158
Coconino	255
Kaibab	166
Lincoln	151
Dixie	22
Carson	116
Rio Grande	11
San Juan	69
Gila	48

The Douglas-fir and white fir data sets are considerably smaller in sample size than ponderosa pine, however they originate from 7 national forests located in Arizona, New Mexico, southern Utah, and southern Colorado. The Lincoln National Forest possesses the majority of the

sample trees for both species. A large portion of these trees were collected by Walter G. Thomson in 1937. Sixty four Douglas-fir sample trees were destructively sampled in 1932 on the Santa Fe National Forest and the observed merchantable height - dbh pairs were recorded on stem analysis graph paper. In 1987, the Timber Management Section of the Southwestern Forest Service Region conducted a destructive sampling validation study for Douglas-fir and white fir. The validation study trees were sampled from the Carson, Santa Fe, Lincoln, Kaibab, and Apache-Sitgreaves National Forests. The Rocky Mountain Forest and Range Experiment Station provided destructive samples for 67 Douglas-fir and 7 white fir trees from the Rio Grande National Forest in southern Colorado. We also obtained data on 20 Douglas-fir and 20 white fir trees from the Dixie National Forest in Utah, collected by Kaibab Forest Products Company in 1989. Tables 3 and 4 indicate the ranges and the frequency of data by dbh classes and total height classes for Douglas-fir and white fir respectively. Tables 5 and 6 indicate the distribution of sample trees from 7 national forests in the Southwest for Douglas-fir and white fir respectively.

Table 3
Frequency of Sample Trees by DBH and Total Height Classes

Douglas-fir

dbh (in.)	Total height (ft)											Total
	30.1 -40	40.1 -50	50.1 -60	60.1 -70	70.1 -80	80.1 -90	90.1 -100	100.1 -110	110.1 -120	120.1 -130	130.1 -	
4.1-6	9	1										10
6.1-8	9	8										17
8.1-10	3	15	12									30
10.1-12	1	13	14	10								38
12.1-14		4	7	14	6	2						33
14.1-16		2	8	13	10	1						34
16.1-18			3	9	16	7	1					36
18.1-20			3	10	20	10	4					47
20.1-22				3	14	13	5	1				36
22.1-24			1	4	9	21	6	3	1			45
24.1-26					5	17	12	2				36
26.1-28				1	1	7	12	2	1			25
28.1-30					1	5	4	2	5			17
30.1-32				1	1	1	6	2	1	1		13
32.1-34							1	4	1	1		7
34.1-36							3	2	4	1		10
36.1-38								2	3	2		7
38.1-40								2	2	1		5
40.1-									2	1	3	6
Total	22	43	48	65	83	84	54	22	21	7	3	452

Table 4
Frequency of Sample Trees by DBH and Total Height Classes
White fir

dbh (in.)	Total height (ft)										Total
	30.1 -40	40.1 -50	50.1 -60	60.1 -70	70.1 -80	80.1 -90	90.1 -100	100.1 -110	110.1 -120	120.1 -130	
6.1-8	1	1									2
8.1-10		5	9	1							15
10.1-12		2	11	9	2						24
12.1-14	2	1	3	25	9	1	1				42
14.1-16			3	30	24	9	2				68
16.1-18			3	10	15	18	6	1			53
18.1-20			1	2	14	15	15				47
20.1-22				1	13	14	9	5			42
22.1-24					2	12	6	5	1	1	28
24.1-26				2		7	7	4	2	2	24
26.1-28				2	2	2	6	9	2	3	26
28.1-30							2	2	3		7
30.1-32							3	1	2	1	7
32.1-34								1	1		2
34.1-36									1		1
36.1-38										1	1
38.1-										1	1
Total	3	9	30	83	81	78	57	28	12	9	390

Table 5
Distribution of Sample Trees from
7 National Forests

Douglas-fir	
National Forest	Number of sample trees
Apache-Sitgreaves	26
Santa Fe	77
Kaibab	37
Lincoln	211
Dixie	20
Carson	14
Rio Grande	67

Table 6
Distribution of Sample Trees from
7 National Forests

White fir	
National Forest	Number of sample trees
Apache-Sitgreaves	21
Santa Fe	12
Kaibab	32
Lincoln	288
Dixie	20
Carson	10
Rio Grande	7

Sample trees that were forked, crooked, or damaged with die back or disease were excluded entirely from the data set in this study. Unlike the Hann and Bare (1978) approach, we believe there is little utility in constructing equations for defected tree form.

Sample trees from the north district of the Kaibab National Forest of the Eagar data set, and of the data set collected by Kaibab Forest Products Company in 1989 were excluded from this study. The sample trees obtained by Kaibab Forest Products Company from the Dixie National Forest are included in this study. The ponderosa pine data set used by McTague and Stansfield (1988) and known as the North Coconino-South Kaibab (NCSK) study was not used in this analysis. The NCSK fails to distinguish sample trees among the two forests and therefore can not be utilized for the construction of forest specific volume equations.

Some of the data sets that originate from studies in the 30's and 40's in the Myers study contain a preponderance of large, old-growth trees of excellent form, and are rarely encountered in the forest today. We decided however not to exclude any of these trees from the study, since it was certain from the inception of the analysis that a weighted least squares analysis would be used to estimate volume. The weighting procedure give less weight to large trees, therefore obviating the need to omit large trees from the study.

Field Measurement

Prior to felling the sample trees, dbh was measured to the nearest 0.1 inch outside bark, 4.5 feet above ground level on the uphill side of the tree. This operation and precision level was conducted identically for all data sets. Total height was measured on the ground to the nearest tenth foot for each tree in all the data sets of this study, except for the Region 3, Douglas-fir and white fir validation study, where the height was measured to the nearest foot. Sample trees in the Eagar study were generally sectioned into 16-foot logs, and two inside bark diameter measurements were taken at each end. Sample trees in the Myers study were divided into 8.25-foot logs above 9.25 feet, and one diameter inside bark to the nearest 0.1 inch measurement was made at each end of the log. Below 9.25 feet, the Myers study contains a diameter inside bark measurement at stump height, 3 feet, and breast height. The Eagar data set contains one bark thickness measurement at each 32-foot interval.

The ponderosa pine sample trees from the Fort Valley Experimental Forest contain inside and outside bark measurements at approximately 4-foot intervals. The ponderosa pine, Douglas-fir, and white fir sample trees felled by Kaibab Forest Products Company on the Dixie National Forest in Utah possess two outside bark diameter measurements and a double bark thickness measurement at 16.5 foot intervals. The sample trees of all species on the Rio Grande National Forest and of ponderosa pine on the San Juan National Forest contain only a single inside bark

diameter measurement for logs that vary in length from 4 feet to 18.5 feet.

The Douglas-fir and white fir sample trees collected in 1937 on the Lincoln National Forest possess one outside and inside bark diameter measurement for logs that vary in length from 8 to 16 feet. The Region 3, Douglas-fir and white fir validation study, contains sample trees that were bucked into lengths that correspond to 10% of the total tree height. Every tree however contains a diameter measurement at 17.3 feet, the height corresponding to the Girard form class measurement. Two outside bark and one inside bark diameter measurements were made at each bolt cut for the Region 3 validation study. The Douglas-fir sample trees collected on the Santa Fe National Forest in the early 1930's were bucked into half-log length and possess one inside bark diameter measurement.

Computation of Observed Butt
Log Cubic Foot Volume

When the length of butt log (first log) exceeds 10 feet, the Smalian formula is not recommended for estimating the butt log cubic foot volume for species that display butt swell. A common butt log volume estimator used in the Pacific Northwest is the Bruce (1982) butt log estimator. The Bruce equation is recommended for general use based on measurements of length and diameters of both ends of the butt log:

$$V_b = 0.005454 L (0.25 d_l^2 + 0.75 d_u^2)$$

where

V_b = butt log cubic foot volume

L = the length of butt log in feet

d_l = diameter at the lower end of the log in inches

d_u = diameter at the upper end of the log in inches.

While, the Bruce estimator was developed for 22 coniferous species and a few hardwood species in the Pacific Northwest Region, there are substantial differences in the ecological growing conditions between the Pacific Northwest and Southwest, and it is unknown how well the Bruce estimator is suited for the Southwest region. We conducted a comparison between the observed volume for butt logs and the estimate of butt volume as computed from the Bruce estimator, and it was found that the Bruce estimator underestimated the butt log volume of southwestern ponderosa pine. Therefore, new butt log volume estimators were developed for southwestern ponderosa pine, Douglas-fir, and white fir in this study.

The total of 1904 butt log measurements from the Myers data set were used to establish the new butt log volume estimator for ponderosa pine. Since all butt logs in the Eagar study were cut at 14.5 or 16.5 foot lengths, the Smalian formula could not be used to estimate the butt log volume, and were therefore omitted from the model of the new ponderosa pine butt log estimator.

Each butt log in the Myers study had several intermediate diameter and corresponding height measurements between the stump and the 17.5 foot section cut. The Smalian formula was used to estimate volume of these intermediate bolts. The summation of the bolt volumes for the butt log constituted the observed volume.

A non-linear least square regression procedure was used to develop the new butt log volume estimate of ponderosa pine in the Southwest. The new butt log model is a function of tree size, butt log length, and end log diameters. The butt log model is expressed as follows:

$$V_b/L = 0.005454 [(\alpha_0 + \alpha_1 L + \alpha_2/D^2H)d_u^2 + (1 - \alpha_0 - \alpha_1 L - \alpha_2/D^2H)d_l^2]$$

where

V_b = butt log cubic foot volume inside bark

L = butt log length in feet

D = dbh of tree in inches

H = total height of tree in feet

d_u = diameter inside bark at the upper end of the butt log in inches

d_l = diameter inside bark at the lower end of the butt log in inches (stump diameter)

α_i = coefficients to be estimated.

The dependent variable is butt log volume weighted by the inverse of butt log length. A plot of the residuals against the predicted values showed a homogeneous distribution for this weighted model. The model was fit with a corrected coefficient of determination of 0.997, and a standard error of 0.1084. All coefficients are significant at the level $p=0.05$. The model can be expressed as V_b alone with the estimated coefficients:

Ponderosa pine butt log estimator

$$V_b = 0.005454 L [(0.6518 + 0.0046 L - 899.0564/(D^2H))d_u^2 + (0.3482 - 0.0046 L + 899.0564/(D^2H))d_l^2]$$

For those ponderosa pine trees where the length of the butt log exceeded 10 feet, the butt log cubic volume was computed with the new butt log volume estimator above. Sample trees from the Eagar data set and from the Dixie National Forest contain butt logs that exceed 10 feet in length and therefore butt log volumes of these trees were computed with the new estimator.

Non-linear least squares regression procedures were also tested for the development of new butt log volume estimators for southwestern Douglas-fir and white fir. Unlike the ponderosa pine model, butt log volume was found to be a function of only log length and end log diameters. The butt log model for both species is expressed as:

$$V_b/L = 0.005454[(\alpha_0 + \alpha_1 L)d_u^2 + (1 - \alpha_0 - \alpha_1 L)d_l^2]$$

This model was fitted to 635 Douglas-fir butt log measurements. Many Douglas-fir sample trees had several intermediate diameter and corresponding height measurements between the stump and the top of the first log. The fitted corrected coefficient of determination for the Douglas-fir butt log equation was 0.994, with a standard error of 0.0961. The fitted model for white fir consisted of 685 butt log measurements and produced a coefficient of determination of 0.991 and a standard error of 0.0771. Butt log volume expressed in terms of V_b alone with estimated coefficients are:

Douglas-fir butt log volume

$$V_b = 0.005454 L [(0.6019 + 0.0087 L)d_u^2 + (0.3981 - 0.0087 L)d_l^2]$$

White fir butt log volume

$$V_b = 0.005454 L [(0.6112 + 0.0081 L)d_u^2 + (0.3888 - 0.0081 L)d_l^2]$$

Computation of Cubic
Foot Log Volume

For those sample trees where the length of the first log is less than 10 feet, the Smalian formula was used for computing cubic foot volume of the butt logs. The new butt log estimators presented above were used for sample trees where the length exceeded 10 feet. For the upper logs and bolts of all trees from all data sets, the Smalian estimator was also used to compute the

cubic log volume while the top log was treated as a cone. The observed volume was recorded with cumulative values of merchantable volume by bolt cuts beginning with the base of the tree up the stem. Ponderosa pine volume was computed for both inside bark and outside bark values with the exception of the Myers data set and the data sets of the Rio Grande and San Juan National Forests where outside bark measurements were omitted. Douglas-fir volume was computed for both inside bark and outside bark values with exception of the Rio Grande data set and the Santa Fe data set collected in the 1930's. White fir volume was computed for both inside and outside bark values with exception of the Rio Grande data set.

Computation of Scribner
Board Foot Log Volume

The Scribner board foot volume of logs (inside bark by definition) was computed using the Staebler (1952) formula. The board volume of a one-foot section of a log is calculated as 1/16 the volume of a 16-foot log:

$$V_{bf,1} = 0.0625 (0.79 d^2 - 2d - 4)$$

where

$V_{bf,1}$ = the volume in board feet of a 1-foot log by
the Scribner rule

d = inside bark diameter in inches at the small
end of log.

The volume of any log may be computed by inserting the small end diameter into the equation above and multiplying the result by the length of the log. As stated by Avery and Burkhart (1983), the exact minimum board width for the Scribner log rule is not definitely known, although it appears to have been 4 inches for at least some log diameters. We computed the cumulative Scribner board foot volume for a tree until the point where the estimate of volume for a log or bolt was less than or equal to zero board feet.

CHAPTER 4

ANALYSIS AND RESULTS

Following the traditional approach that recognizes the need to discriminate distinct volume equations among blackjack and yellow pine, or to distinguish volume equations based on some threshold value of D^2H , the splining technique outlined by Gallant and Fuller (1973) was applied to construct a segmented volume equation. By using this technique, two or more segments can be joined into a smooth and continuous function. The join point was estimated in the non-linear regression procedure as a function of the combined variable D^2H . At the join point, the estimate of volume is identical for both segments.

It was hypothesized that total tree volume could be estimated with the following model:

$$\begin{aligned} V_1 &= a_0 + a_1 D^2H + a_2 (D^2H)^2 & 0 \leq D^2H \leq J \\ V_2 &= b_0 + b_1 D^2H & J < D^2H < \infty \end{aligned}$$

where V_1, V_2 = total tree volume in cubic feet inside
bark

D = dbh in inches

H = total tree height in feet

J = join point

a_i, b_i = regression coefficients.

A continuous function is insured by conditioning the first derivative of both segments to be equal at the join point. Since both segments predict identical volume and are continuous at the join point, the two segments are algebraically manipulated into a single equation. The following steps are needed to produce a single equation:

Step 1; equate both equations at the join point

$$a_0 + a_1 J + a_2 J^2 = b_0 + b_1 J$$

Step 2; equate the first derivative of both equations at the join point

$$a_1 + 2a_2 J = b_1$$

Step 3; insert the expression for b_1 above into the equation produced from step 1

$$b_0 = a_0 - a_2 J^2$$

Step 4; insert the new expressions for b_0 and b_1 into the volume equation, V_2

$$V_2 = a_0 - a_2 J^2 + a_1 D^2H + 2a_2 J (D^2H)$$

Step 5; by adding and subtracting the term, $a_2(D^2H)^2$, to the equation above, it is possible to express V_2 as:

$$V_2 = V_1 - a_2 (J - D^2H)^2 I$$

or

$$V_2 = a_0 + a_1 (D^2H) + a_2(D^2H)^2 - a_2 (J - D^2H)^2 I$$

where

$$I = 1 \quad \text{if} \quad J < D^2H$$

$$I = 0 \quad \text{if} \quad J \geq D^2H$$

This model was then fitted individually to ponderosa pine sample tree data from each national forest. The parameters of this model were estimated with a non-linear least squares procedure. Since the ponderosa pine sample sizes from the Dixie, Rio Grande, San Juan and Gila National Forests were quite small, these data sets were initially excluded from the fitting procedure. The estimated coefficients and join point are listed in Table 7.

The results from Table 7 show that the coefficients are very different among national forests. The join point varied drastically from one national forest to another. Some join points are out of the available data range, and most of the join points possess a very small value of D^2H , which represents a very small tree. Therefore, there is

Table 7
Regression Coefficients, Join Point and Observed
D²H Range of the Segmented Polynomial Total Volume
Equations for Ponderosa Pine by National Forest

National Forest	a ₀	a ₁	a ₂	J	Observed D ² H range
Tonto	0.2803	0.0019	0	146214	0 - 140000
A/S	0.1248	0.0019	0	43639.81	0 - 250000
Sfe.	-2.7550	0.0024	0	3577.82	0 - 250000
Prs.	95.147	-0.1252	0.00004	1588.15	0 - 250000
Coc.	0.4329	0.0019	0	38541.17	0 - 120000
Kai.	-0.0513	0.0020	0	74105.27	0 - 250000
Lin.	5.4666	0.0014	0	-1466.39	0 - 175000
Car.	-0.1091	0.0018	0	-23483.6	0 - 150000

Note: A/S = Apache-Sitgreaves National Forest
 Sfe. = Santa Fe National Forest
 Prs. = Prescott National Forest
 Coc. = Coconino National Forest
 Kai. = Kaibab National Forest
 Lin. = Lincoln National Forest
 Car. = Carson National Forest

little apparent need to distinguish the estimation between young growth and old growth trees for ponderosa pine. Although this result may appear to contradict earlier studies by Hornibrook (1936) and Hann and Bare (1978), this conclusion was reached after imposing the conditions of steps 1 and 2 above on the young growth and old growth equation. The splining procedure implicitly removes the independence of the two segments. In concordance with the Hann and Bare approach, we did not attempt to make distinct models for young growth and old growth Douglas-fir and white fir.

Empirical Bayes Estimator

The application of Empirical Bayes (EB) estimation to the linear model has been studied by many authors (Zellner and Vandaele 1975, Morris 1983). They have usually studied the application of the EB estimation to one linear model, in which all the individual coefficients from a linear regression are shrunk toward some center point. Thus, individual coefficients are estimated simultaneously. While it can be shown that this estimator dominates the usual least squares estimator under quadratic loss, the gains are unlikely to be large unless the individual coefficients in the model are similar (Green et al. 1990).

ETS Estimator

The Educational Testing Service (ETS) investigated EB estimation techniques to develop models to predict graduate school performance of students based on their undergraduate grade point average and GRE score. Unique models were required for each of the 190 graduate departments included in the study. However, it was determined that for many departments there was insufficient data to construct a reliable model. The solution was to develop the following Bayesian structure for the problem.

The first stage model is given by

$$Y_i = X_i\beta_i + \epsilon_i \quad i = 1, 2, \dots, m \quad (1)$$

where Y_i is an $(n_i \times 1)$ vector of dependent variables, X_i is an $(n_i \times p)$ design matrix containing values for the p independent variables for each of the n_i students in department i , β_i is a $(p \times 1)$ vector of regression coefficients for department i , and ϵ_i is an $(n_i \times 1)$ vector of $N(0, \sigma^2)$ random errors, and m is the number of departments.

The second stage model is given by

$$\beta = Z\Gamma + \Delta \quad (2)$$

where β is a $(m \times p)$ matrix of regression coefficients, Z is a $(m \times k)$ matrix of dependent covariates, Γ is a $(k \times p)$ matrix of second-stage regression coefficients, Δ is a $(m \times p)$ matrix of random errors where the i^{th} row of Δ is denoted Δ_i' , and $\Delta_i \sim N(0, \Sigma)$.

Model (1) and (2) constitute a Bayesian model. According to this model, the vectors of the true regression coefficients are themselves generated from a regression plane characterized by the matrix of hyperparameters Γ and independent normal deviations distributed according to the variance-covariance matrix Σ . Model (2) captures all the systematic variation between β_i 's.

Models (1) and (2) imply that β_i 's follow a joint multivariate normal distribution with mean:

$$(\Sigma^{-1} + \sigma_i^{-2} X_i' X_i)^{-1} (\Sigma^{-1} \Gamma' Z_i + \sigma_i^{-2} X_i' X_i \beta_i)$$

Estimation of β_i is straight forward using ordinary linear regression methods. The σ_i^2 , Σ , and Γ were estimated by the EB algorithm developed by Dempster et al. (1977). The algorithm is iterative and consists of an estimation (E) step in which conditional values of the parameters are determined, followed by a maximization (M) step during which the likelihood function is maximized based on the conditional values of the parameters.

Application to Volume Models

We used the EB procedure developed by ETS to construct volume equations for southwestern ponderosa pine, Douglas-fir, and white fir. One unique model for each measure of volume, whether cubic foot or board foot, total or merchantable, was constructed for each of the national forests. In the case of the total volume inside bark model for ponderosa pine, the covariates selected were an intercept and a dummy variable which indicate the presence or absence on the Kaibab, Coconino, Apache-Sitgreaves, Gila or Lincoln National Forests. Thus, the second-stage (hyperparameter) model was:

$$b_i' = Z'\Gamma + \Delta_i$$

where b_i' is a $(1 \times p)$ row vector of regression coefficients for national forest i , p is number of independent variables in the first-stage model including the intercept, Z' is (1×2) covariate vector, Γ is a $(2 \times p)$ matrix of hyperparameters, and Δ_i is a $(1 \times p)$ error row vector.

Total Volume

Total Inside Bark Cubic Foot Volume Equation

Model Construction Several published volume equations for southwestern ponderosa pine, Douglas-fir, and white fir

have applied weights or transformations to adjust for the increasing stem content variability of large trees (Minor 1961, Hornibrook 1936, Peterson 1958, Hann and Bare 1978). Recognizing the need for a weighting procedure, the following least squares model was used to predict total tree cubic foot volume:

$$\ln V_{t,ib} = b_0 + b_1 \ln D + b_2 \ln H \quad (3)$$

where $\ln$ = the natural logarithm function
 $V_{t,ib}$ = total inside bark cubic foot volume
 D = dbh in inches
 H = total height in feet
 b_i = regression coefficients.

Equation (3) can be expressed in non-linear form as follows:

$$V_{t,ib} = a_0 D^{b_1} H^{b_2} \quad (4)$$

where $a_0 = e^{b_0}$ and e is the base of natural logarithms.

Equation (3) was fitted to the data from each national forest by species using ordinary least squares (OLS). Sample trees that possessed a dbh below 2 inches and certain outliers that contained obvious field measurement errors

were eliminated from the data set. The estimated regression coefficients by national forest are listed in Tables 8, 9, and 10 for ponderosa pine, Douglas-fir, and white fir respectively. The results by species show that the coefficients differ among national forests. We did not pool the sample trees by species and construct a single volume equation for the entire southwestern region because this approach would fail to reflect some possible differences among the diverse forests and sub-regions. While ideally, one could rely on volume equations that are parameter specific to each national forest, the small sample size for some national forests implies some potential problems of precision and extrapolation. The EB methodology provides a good solution that depends on both the precision of the within-forest estimate and among-forest relationship.

Modeling of Hyperparameters Several hyperparameter models were made for ponderosa pine exploring the use of covariates such as rainfall, elevation, and location. Based on geographic location, it is known that the Kaibab, Coconino, Apache-Sitgreaves, and Gila National Forests are on the Mogollon Rim at approximately 7000 ft in elevation, and the Prescott and Tonto National Forests are below the rim. East of the Mogollon Rim, an escarpment forming the southern boundary of the Colorado plateau, there is a distinct winter rain shadow (Jameson 1969). In Arizona

Table 8
Coefficients for the Total Inside Bark Cubic Foot
Volume Equation from OLS Regression
by National Forest

Ponderosa Pine						
National Forest	b_0	b_1	b_2	Root MSE	R^2	No. of sample trees (N)
Tonto	-6.5982	2.1189	1.0162	0.1152	0.9875	64
A/S	-6.2494	1.9924	1.0244	0.1462	0.9915	653
Santa Fe	-6.5347	2.0563	1.0553	0.1046	0.9924	166
Prescott	-6.4715	2.1034	0.9707	0.1255	0.9878	158
Coconino	-6.4274	1.9760	1.0749	0.1201	0.9959	255
Kaibab	-6.5951	2.0170	1.0792	0.1139	0.9906	166
Lincoln	-6.2830	1.9271	1.0799	0.1014	0.9879	151
Dixie	-6.9168	2.1566	1.0724	0.1251	0.9809	22
Carson	-6.3772	2.1149	0.9533	0.1141	0.9884	116
R. Grande	-6.1164	2.1527	0.9876	0.0643	0.9946	11
San Juan	-5.6920	2.0381	0.8809	0.0897	0.9883	69
Gila	-6.1460	1.9395	1.0450	0.1223	0.0744	48

Note: A/S = Apache-Sitgreaves National Forest
 R.Grande = Rio Grande National Forest
 Root MSE is the estimated standard error of the model
 R^2 is the coefficient of determination

The least squares linear regression model for total volume is as follows:

$$\ln V_{t,ib} = b_0 + b_1 \ln D + b_2 \ln H$$

where

$V_{t,ib}$ = total inside bark cubic foot volume
 D = dbh in inches
 H = total height in feet.

Table 9
Coefficients for the Total Inside Bark Cubic Foot
Volume Equation from OLS Regression
by National Forest

Douglas-fir						
National Forest	b_0	b_1	b_2	Root MSE	R^2	No. of sample trees (N)
A/S	-8.5840	1.2941	2.0041	0.0735	0.9939	26
Santa Fe	-6.4905	1.6652	1.2730	0.1039	0.9719	77
Kaibab	-5.4357	1.5573	1.1130	0.1189	0.9717	37
Lincoln	-6.0164	1.8185	1.0455	0.1137	0.9936	211
Dixie	-6.3040	1.6072	1.2941	0.0787	0.9910	20
Carson	-7.3492	1.5150	1.5756	0.0892	0.9830	14
R. Grande	-5.0428	1.8817	0.8591	0.1433	0.9705	67

Note: A/S = Apache-Sitgreaves National Forest
 R.Grande = Rio Grande National Forest
 Root MSE is the estimated standard error of the model
 R^2 is the coefficient of determination

The least squares linear regression model for total volume is as follows:

$$\ln V_{t,ib} = b_0 + b_1 \ln D + b_2 \ln H$$

where

$V_{t,ib}$ = total inside bark cubic foot volume
 D = dbh in inches
 H = total height in feet.

Table 10
Coefficients for the Total Inside Bark Cubic Foot
Volume Equation from OLS Regression
by National Forest

White fir						
National Forest	b_0	b_1	b_2	Root MSE	R^2	No. of sample trees (N)
A/S	-6.4917	1.6132	1.3145	0.0729	0.9920	21
Santa Fe	-6.9328	1.6796	1.3602	0.0899	0.9888	12
Kaibab	-3.9847	1.7191	0.6938	0.1731	0.9425	32
Lincoln	-5.8592	1.8628	0.9706	0.1316	0.9702	288
Dixie	-6.0104	1.8816	1.0429	0.0801	0.9875	20
Carson	-4.2667	1.6467	0.7664	0.1500	0.9567	10
R. Grande	-6.3858	1.8052	1.2587	0.0776	0.9860	7

Note: A/S = Apache-Sitgreaves National Forest
 R.Grande = Rio Grande National Forest
 Root MSE is the estimated standard error of the model
 R^2 is the coefficient of determination

The least squares linear regression model for total volume is as follows:

$$\ln V_{t,ib} = b_0 + b_1 \ln D + b_2 \ln H$$

where

$V_{t,ib}$ = total inside bark cubic foot volume
 D = dbh in inches
 H = total height in feet.

and west-central New Mexico, the effect of the summer Sonoran Monsoon decreases rapidly toward the north, suggesting that the Lincoln National Forest displays climatological similarities to the Mogollon Rim. The Dixie, San Juan, Rio Grande, Carson, and Santa Fe National Forests are little influenced by the monsoon air masses, and much of the summer rainfall is a result of convective storms. In Table 8, it is apparent that the estimated coefficients b_1 and b_2 corresponding to $\ln D$ and $\ln H$ respectively for ponderosa pine are very close in values among those national forests on the Mogollon Rim, including the Lincoln National Forest. Therefore, for the purpose of predicting the ponderosa pine parameters b_1 and b_2 of the first-stage with a hyperparameter model, the national forests were divided into two groups. The first group consists of the Kaibab, Coconino, Apache-Sitgreaves, Gila, and Lincoln National Forests, while the other group consists of the Prescott, Tonto, Dixie, Rio Grande, San Juan, Carson, and Santa Fe National Forests. No distinction by national forest was used in the ponderosa pine hyperparameter model for the intercept b_0 . In the case of Douglas-fir and white fir, we were unable to detect any differences among the b_0 , b_1 , or b_2 coefficients in Tables 9 and 10 by national forest depending on location of the forest in relation to the Mogollon Rim.

We modeled the estimated coefficients of the first-stage with a hyperparameter model using seemingly unrelated

regression (Zellner 1962). The ponderosa pine hyperparameter model was given by:

$$b_i' = Z_i' \Gamma + \Delta_i \quad (5)$$

where

$Z_i' =$ a (1 x 2) vector, $[1, Z_1]$

$$\begin{cases} Z_1 = 1 & \text{if sample tree is on the Apache-} \\ & \text{Sitgreaves, Kaibab, Coconino,} \\ & \text{Gila, and Lincoln National Forests.} \\ Z_1 = 0 & \text{otherwise} \end{cases}$$

$b_i' =$ a (1 x 3) vector of regression coefficients

for national forest i

$\Gamma =$ a (2 x 3) matrix of coefficients of
hyperparameters

$\Delta_i =$ a (1 x 3) error vector.

The ponderosa pine coefficients from joint generalized least squares regression, or seemingly unrelated regression are listed in Table 11. This procedure estimates the hyperparameters b_0 , b_1 , and b_2 simultaneously and takes into account cross-equation correlation. The predictor variable of the ponderosa pine total inside bark cubic foot volume hyperparameter model, equation (5), produced an insignificant coefficient in the case of total inside bark cubic foot volume for Douglas-fir and white fir. Therefore, the covariate selected for modeling the hyperparameters of these two species

Table 11
Estimated Hyperparameter Coefficients of
Equation (5) and Associated t-test

Ponderosa Pine

Dependent variable (b_i)	Independent variable	Df	Parameter estimates (Γ)	Standard error	t for $H_0: \Gamma_i = 0$	Prob > T
b_0	intercept	1	-6.3673	0.0887	-71.77	0.0001
b_1	intercept	1	2.1045	0.0148	142.18	0.0001
	Z_1	1	-0.1321	0.0204	-6.469	0.0001
b_2	intercept	1	0.9881	0.0166	59.387	0.0001
	Z_1	1	0.0766	0.0161	4.766	0.0008

Note: The Γ (2 x 3) matrix may be expressed with the estimated coefficients as:

$$\Gamma = \begin{bmatrix} -6.3673 & 2.1045 & 0.9881 \\ 0 & -0.1321 & 0.0766 \end{bmatrix}$$

was the unit scaler, which implies that the Empirical Bayes estimate, $\tilde{B}_i$, shrinks toward a point. The model was as follows:

$$b_i' = Z \Gamma + \Delta_i \quad (6)$$

where $b_i' = a$ (1 x 3) vector of regression coefficients
from OLS estimation for national forest i.

Z = an unit scaler.

Γ = is a (1 x 3) vector of coefficients of hyperparameters.

Δ_i = a (1 x 3) error vector.

The coefficients estimated from seemingly unrelated regression for Douglas-fir and white fir are listed in Tables 12 and 13 respectively.

Table 12
Estimated Hyperparameter coefficients of
Equation (6) and Associated t-test

Douglas-fir

Dependent variable (b_i)	Independent variable	Df	Parameter estimates (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> T $
b_0	Z	1	-6.4604	0.4520	-14.29	0.0001
b_1	Z	1	1.6199	0.0742	21.83	0.0001
b_2	Z	1	1.3092	0.1436	9.118	0.0001

Note: Z = unit scaler

Table 13
Estimated Hyperparameter coefficients of
Equation (6) and Associated t-test

White fir

Dependent variable (b_i)	Independent variable	Df	Parameter estimates (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> T $
b_0	Z	1	-5.7045	0.4292	-13.29	0.0001
b_1	Z	1	1.7440	0.0403	43.30	0.0001
b_2	Z	1	1.0582	0.1004	10.54	0.0001

Note: Z = unit scaler

EB Estimator Once the hyperparameters were modeled, the EB estimate was computed with the following equation:

$$\tilde{B}_i = \frac{\Sigma^{-1}\Gamma'Z + (1/MSE_i)(X_i'X_i)b_i}{\Sigma^{-1} + (1/MSE_i)(X_i'X_i)} \quad (7)$$

where $\tilde{B}_i$ = a (3 x 1) vector of estimated EB coefficients for national forest i

Σ = a (3 x 3) variance-covariance matrix of the hyperparameters

MSE_i = mean square error from the OLS estimate for national forest i

Γ = a (2 x 3) matrix of hyperparameters for ponderosa pine and a (1 x 3) vector of hyperparameters for Douglas-fir and white fir

Z = a (2 x 1) covariate vector for ponderosa pine and a unit scaler for Douglas-fir and white fir

b_i = a (3 x 1) vector of regression coefficients from the OLS estimation for national forest i

$X_i'X_i$ = crossproducts prediction variables matrix for national forest i.

The estimated EB posterior variance-covariance matrix is expressed as:

$$\text{Var} = [\Sigma^{-1} + (\text{MSE } (X_i' X_i)^{-1})^{-1}]^{-1} \quad (8)$$

The following is a example of an EB estimate computation for ponderosa pine on the Tonto National Forest:

$$\tilde{B}_i =$$

$$\begin{aligned}
 & \left[\begin{array}{ccc} 52.7957 & 109.7269 & 230.9262 \\ 109.7269 & 882.2131 & 290.9804 \\ 230.9262 & 290.9804 & 1423.8520 \end{array} \right] + \\
 & \quad (X_i' X_i) \\
 & \quad (1/\text{MSE}) \left[\begin{array}{ccc} 64 & 183.1996 & 274.9241 \\ 183.1996 & 534.5354 & 791.1056 \\ 274.9241 & 791.1056 & 1183.4580 \end{array} \right]^{-1} * \\
 & \quad (1/0.01327) * \left[\begin{array}{ccc} 52.7957 & 109.7269 & 230.9262 \\ 109.7269 & 882.2131 & 290.9804 \\ 230.9262 & 290.9804 & 1423.8520 \end{array} \right] \left[\begin{array}{cc} -6.3673 & 0 \\ 2.1045 & -0.132 \\ 0.9881 & 0.0766 \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \\
 & \quad + (1/0.01327) \left[\begin{array}{ccc} 64 & 183.1996 & 274.9241 \\ 183.1996 & 534.5354 & 791.1056 \\ 274.9241 & 791.1056 & 1183.4580 \end{array} \right] \left[\begin{array}{c} -6.5982 \\ 2.1189 \\ 1.0162 \end{array} \right]
 \end{aligned}$$

The EB estimate for other forests and species followed the procedure above, and the ponderosa pine coefficients of the EB estimates for 12 national forests are listed in Table 14. The EB estimates of 7 national forests for Douglas-fir and white fir are listed in Tables 15 and 16 respectively. The coefficients of Tables 14-16 may be transformed and inserted into non-linear volume equations. The non-linear total inside bark cubic foot volume equations are presented in Tables 17-19.

Table 14
Parameters of the EB Estimate for the Total Inside Bark
Cubic Foot Volume Equation

Ponderosa Pine			
National Forest	$\tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$
Tonto	-6.5280	2.1150	1.0027
A/S	-6.2835	1.9788	1.0417
Santa Fe	-6.4453	2.0904	1.0111
Prescott	-6.4961	2.0940	0.9833
Coconino	-6.4216	1.9770	1.0728
Kaibab	-6.6103	2.0020	1.0934
Lincoln	-6.2557	1.9517	1.0565
Dixie	-6.8592	2.0647	0.9807
Carson	-6.4230	2.0973	0.9765
Rio Grande	-6.0240	2.1262	0.9793
San Juan	-5.8693	2.0487	0.9152
Gila	-6.2205	1.9617	1.0480

Note: the model for the EB estimate is as follows:

$$\ln V_{t,ib} = \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln H$$

Table 15
Parameters of the EB Estimate for the Total Inside
Bark Cubic Foot Volume Equation

Douglas-fir			
National Forest	$\tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$
A/S	-8.0719	1.3989	1.8134
Santa Fe	-6.5172	1.6417	1.2957
Kaibab	-5.7489	1.6008	1.1563
Lincoln	-6.0721	1.8004	1.0707
Dixie	-6.2592	1.6268	1.2701
Carson	-7.0392	1.5357	1.4886
Rio Grande	-4.9524	1.8724	0.8425

The model for the EB estimate is as follows:

$$\ln V_{t,ib} = \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln H$$

Table 16
Parameters of the EB Estimate for the Total Inside
Bark Cubic Foot Volume Equation

White fir			
National Forest	$\tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$
A/S	-6.0233	1.7063	1.1419
Santa Fe	-6.5920	1.7460	1.2390
Kaibab	-4.3996	1.7072	0.7969
Lincoln	-5.9053	1.8401	0.9961
Dixie	-6.1284	1.8155	1.1146
Carson	-5.1233	1.6932	0.9380
Rio Grande	-6.0763	1.7876	1.1897

The model for the EB estimate is as follows:

$$\ln V_{t,ib} = \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln H$$

Table 17
The Total Inside Bark Cubic Foot
Volume ($V_{t.ib}$) Equations

Ponderosa Pine

National Forest	Equation
Tonto	$V_{t.ib} = 0.001462 D^{2.1150} H^{1.0027}$
Apache-Sitgreaves	$V_{t.ib} = 0.001867 D^{1.9788} H^{1.0417}$
Santa Fe	$V_{t.ib} = 0.001588 D^{2.0904} H^{1.0111}$
Prescott	$V_{t.ib} = 0.001509 D^{2.0940} H^{0.9833}$
Coconino	$V_{t.ib} = 0.001626 D^{1.9770} H^{1.0728}$
Kaibab	$V_{t.ib} = 0.001346 D^{2.0020} H^{1.0934}$
Lincoln	$V_{t.ib} = 0.001919 D^{1.9517} H^{1.0565}$
Dixie	$V_{t.ib} = 0.001050 D^{2.0647} H^{0.9807}$
Carson	$V_{t.ib} = 0.001624 D^{2.0973} H^{0.9765}$
Rio Grande	$V_{t.ib} = 0.002420 D^{2.1262} H^{0.9793}$
San Juan	$V_{t.ib} = 0.002825 D^{2.0487} H^{0.9152}$
Gila	$V_{t.ib} = 0.001988 D^{1.9617} H^{1.0480}$

Table 18
The Total Inside Bark Cubic Foot
Volume ($V_{t,ib}$) Equations

Douglas-fir

National Forest	Equation
Apache-Sitgreaves	$V_{t,ib} = 0.000312 D^{1.3989} H^{1.8134}$
Santa Fe	$V_{t,ib} = 0.001478 D^{1.6417} H^{1.2957}$
Kaibab	$V_{t,ib} = 0.003186 D^{1.6008} H^{1.1563}$
Lincoln	$V_{t,ib} = 0.002306 D^{1.8004} H^{1.0707}$
Dixie	$V_{t,ib} = 0.001913 D^{1.6268} H^{1.2701}$
Carson	$V_{t,ib} = 0.000877 D^{1.5357} H^{1.4886}$
Rio Grande	$V_{t,ib} = 0.007067 D^{1.8724} H^{0.8425}$

Table 19
The Total Inside Bark Cubic Foot
Volume ($V_{t.ib}$) Equations

White fir

National Forest	Equation
Apache-Sitgreaves	$V_{t.ib} = 0.002422 D^{1.7063} H^{1.1419}$
Santa Fe	$V_{t.ib} = 0.001371 D^{1.7460} H^{1.2390}$
Kaibab	$V_{t.ib} = 0.012283 D^{1.7072} H^{0.7969}$
Lincoln	$V_{t.ib} = 0.002725 D^{1.8401} H^{0.9961}$
Dixie	$V_{t.ib} = 0.002180 D^{1.8155} H^{1.1146}$
Carson	$V_{t.ib} = 0.005956 D^{1.6932} H^{0.9380}$
Rio Grande	$V_{t.ib} = 0.002297 D^{1.7876} H^{1.1897}$

Total Outside Bark Cubic
Foot Volume Equation

Since only a very small number of the ponderosa pine sample trees contained outside bark cubic foot volume information, total outside bark cubic foot volume equations were not made for specific national forests. Rather, one equation was developed for all national forests. The Douglas-fir and white fir data sets contained considerably more outside bark observations, however neither data set contained outside bark information on the Rio Grande National Forest. Therefore, we resolved to construct southwestern regional outside bark volume equations by species, where total outside

bark cubic foot volume is a function of total inside bark cubic foot volume. The model has the following form:

$$V_{t.ob} = b_0 V_{t.ib}^{b_1} \quad (9)$$

Model (9) can be linearized by taking the natural logarithm function as follows:

$$\ln V_{t.ob} = \ln b_0 + b_1 \ln V_{t.ib} \quad (10)$$

where

$V_{t.ob}$ = total outside bark cubic foot volume

$V_{t.ib}$ = total inside bark cubic foot volume.

A total of 186 ponderosa pine sample trees were fitted to this model by ordinary linear least squares regression. The equation with estimated coefficients is expressed as:

$$\ln V_{t.ob} = 0.3625 + 0.9671 \ln V_{t.ib} \quad (11)$$

or

Ponderosa Pine

$$V_{t.ob} = 1.4369 V_{t.ib}^{0.9671} \quad (12)$$

The coefficient of determination for equation (11) exceeds 0.999. Likewise, equation (10) was fitted to 319 Douglas-fir sample trees and 383 white fir sample trees. Their coefficient of determinations exceed 0.998 and 0.989 respectively. The total volume outside bark functions for Douglas-fir and white fir expressed in non-linear form are:

Douglas-fir

$$V_{t.ob} = 1.3238 V_{t.ib}^{0.9974}$$

White fir

$$V_{t.ob} = 1.3343 V_{t.ib}^{1.0014}$$

Total Scribner Board Foot Volume Equation

Model Selection Hann and Bare (1978) developed an equation to predict International 1/4 inch board foot volume for commercial southwestern tree species. Their equation was as follows:

$$V_I = R_{1/c} V_m \quad (13)$$

where

V_I = International 1/4 inch board foot volume

V_m = predicted gross merchantable cubic foot
volume to a 6 inch top

$R_{V/c}$ = ratio of V_I to V_m

$$R_{V/c} = d_0 - d_1/D - d_2/D^2 - d_3/D^3$$

where $d_0 - d_3$ are species and national forest
specific coefficients
 D = dbh in inches.

Similar to the Hann and Bare concept of constructing a total volume equation for board foot volume, we inserted the independent variable $V_{t,ib}$ into the board foot volume (equation 14). We then compared its performance to the traditional logarithmic standard volume equation (equation 15). Equations 14 and 15 were as follows:

$$\ln V_{t,bf} = b_0 + b_1 \ln D + b_2 \ln H + b_3 \ln V_{t,ib} \quad (14)$$

$$\ln V_{t,bf} = b_0 + b_1 \ln D + b_2 \ln H \quad (15)$$

where

$V_{t,bf}$ = total Scribner board foot volume.

Ponderosa pine sample tree data from each national forest were fitted to these two models with ordinary least squares regression. The results revealed that the model with $V_{t,ib}$

(total inside bark cubic foot volume) gave insignificant parameter estimates for b_0 , b_1 , and b_2 , and only the coefficient associated with $V_{t,ib}$ was significant. Since the prediction equation for $V_{t,ib}$ was based on D and H , it seems logical that $V_{t,ib}$ is highly correlated to the independent variables D and H . Therefore, equation (14) displays the problem of multicollinearity. However, model (15) without the $V_{t,ib}$ term provided significant estimates of the parameters b_0 , b_1 , and b_2 for ponderosa pine on all national forests, and the coefficients of determination (R^2) were very high. Thus, equation (15) was selected to predict Scribner board foot volume.

Model (15) can be expressed in non-linear form after transformation as:

$$V_{t,bf} = e^{b_0 D^{b_1} H^{b_2}} \quad (16)$$

The regression coefficients and summary statistics for ponderosa pine, Douglas-fir, and white fir by national forest are listed in Tables 20-22.

Modeling Hyperparameters The predictor variable of the ponderosa pine total inside bark cubic foot volume hyperparameter model, equation (5), produced an insignificant coefficient in the case of Scribner board foot volume. Therefore, equation (6) was selected for modeling the

Table 20

Coefficients for the Total Scribner Board Foot Volume
Equation from OLS Regression by National Forest

Ponderosa Pine

National Forest	b_0	b_1	b_2	Root MSE	R^2	Number of sample trees (N)
Tonto	-7.2353	3.0062	0.9658	0.2107	0.9768	64
A/S	-8.9161	2.7407	1.5190	0.3076	0.9741	631
Santa Fe	-6.9043	2.4285	1.2887	0.1710	0.9855	166
Prescott	-6.4239	2.8149	0.8609	0.2292	0.9738	158
Coconino	-8.3038	3.6381	0.7377	0.4118	0.9377	225
Kaibab	-6.9953	2.6803	1.1078	0.2723	0.9662	166
Lincoln	-6.7501	2.3001	1.3417	0.1701	0.9770	151
Dixie	-7.2740	2.5578	1.2687	0.1858	0.9673	22
Carson	-6.6779	2.9159	0.8747	0.2217	0.9732	116
R. Grande	-8.8291	2.8607	1.5403	0.1697	0.9740	11
San Juan	-6.0280	2.3399	1.1358	0.1364	0.9806	69
Gila	-7.3121	2.4661	1.3425	0.1937	0.9593	48

Note: A/S = Apache-Sitgreaves National Forest
R. Grande = Rio Grande National Forest

The model to predict total Scribner board foot volume is as follows:

$$\ln V_{t,bf} = b_0 + b_1 \ln D + b_2 \ln H$$

Table 21
Coefficients for the Total Scribner Board Foot Volume
Equation from OLS Regression by National Forest

Douglas-fir

National Forest	b_0	b_1	b_2	Root MSE	R^2	Number of sample trees (N)
A/S	-9.8651	1.4596	2.5910	0.1260	0.9878	26
Santa Fe	-5.7279	1.9349	1.3244	0.1304	0.9649	77
Kaibab	-7.0503	1.9015	1.6386	0.1551	0.9711	37
Lincoln	-9.7892	2.2605	1.9685	0.3520	0.9701	211
Dixie	-7.7914	2.0671	1.7039	0.1250	0.9865	20
Carson	-7.2000	1.9320	1.6667	0.1225	0.9788	14
R. Grande	-6.2349	2.1507	1.3300	0.2085	0.9590	67

Note: A/S = Apache-Sitgreaves National Forest
R. Grande = Rio Grande National Forest

The model to predict total Scribner board foot volume is as follows:

$$\ln V_{t,bf} = b_0 + b_1 \ln D + b_2 \ln H$$

Table 22

Coefficients for the Total Scribner Board Foot Volume
Equation from OLS Regression by National Forest

White fir

National Forest	b_0	b_1	b_2	Root MSE	R^2	Number of sample trees (N)
A/S	-6.0766	2.1348	1.2696	0.1278	0.9836	21
Santa Fe	-6.9503	1.9863	1.5601	0.1186	0.9858	12
Kaibab	-6.6254	2.2542	1.3245	0.1252	0.9847	32
Lincoln	-6.1768	2.2198	1.1829	0.1993	0.9532	288
Dixie	-7.2656	2.2487	1.4621	0.1137	0.9839	20
Carson	-3.4762	2.0456	0.7165	0.2169	0.9386	10
R. Grande	-6.2814	1.9444	1.5001	0.0550	0.9945	7

Note: A/S = Apache-Sitgreaves National Forest
R. Grande = Rio Grande National Forest

The model to predict total Scribner board foot volume is as follows:

$$\ln V_{t,bf} = b_0 + b_1 \ln D + b_2 \ln H$$

hyperparameters, which implies that the Empirical Bayes estimate, $\bar{B}_i$, shrinks toward a point. The model was as follows:

$$b_i' = Z \Gamma + \Delta_i \quad (17)$$

where b_i' = a (1 x 3) vector of regression coefficients
from OLS estimation for national forest i.

Z = an unit scaler.

Γ = is a (1 x 3) vector of coefficients of
hyperparameters.

Δ_i = a (1 x 3) error vector.

The coefficients estimated from seemingly unrelated regression by species are listed in Tables 23-25.

Table 23
Estimated Hyperparameter Coefficients of
Equation (17) and Associated t-test

Ponderosa Pine

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
b_0	Z	1	-7.3042	0.2652	-27.538	0.0001
b_1	Z	1	2.7291	0.1062	25.6910	0.0001
b_2	Z	1	1.1653	0.0758	15.3770	0.0001

Note: Z = unit scaler

Table 24
Estimated Hyperparameter Coefficients of
Equation (17) and Associated t-test

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
b_0	Z	1	-7.6655	0.6126	-12.514	0.0001
b_1	Z	1	1.9580	0.0968	20.233	0.0001
b_2	Z	1	1.7462	0.1643	10.627	0.0001

Note: Z = unit scaler

Table 25
Estimated Hyperparameter Coefficients of
Equation (17) and Associated t-test

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
b_0	Z	1	-6.1218	0.4700	-13.025	0.0001
b_1	Z	1	2.1191	0.0485	43.660	0.0001
b_2	Z	1	1.2880	0.1079	11.940	0.0001

Note: Z = unit scaler

EB Estimate The EB estimate for Scribner board foot volume followed the procedure used previously in the total inside bark cubic foot volume estimation, and the parameters of the EB estimate by species are listed in Tables 26-28. The non-linear total Scribner board foot volume equations for each national forest by species are presented in Tables 29-31.

Merchantable Volume and Taper Function

Merchantable Inside Bark Cubic Foot Volume Equations

Model Selection The ponderosa pine merchantable volume equations presented below predict stem content to any upper-stem diameter or any merchantable height, and are slight modifications to the Burkhart (1977) and Modified Burkhart

Ratio Models. The following models are conditioned so that merchantable volume equals total volume when merchantable

Table 26
Parameters of the EB Estimate for the Total
Scribner Board Foot Volume Equation

Ponderosa Pine

National Forest	$\bar{B}_0$	$\bar{B}_1$	$\bar{B}_2$
Tonto	-7.5548	2.9552	1.0737
A/S	-8.7832	2.7756	1.4645
Santa Fe	-6.9039	2.4336	1.2850
Prescott	-6.5606	2.7850	0.9138
Coconino	-8.3324	3.5571	0.8032
Kaibab	-6.9891	2.6819	1.1056
Lincoln	-6.7473	2.3125	1.3323
Dixie	-7.0575	2.5991	1.1897
Carson	-6.8253	2.8799	0.9340
Rio Grande	-7.6073	2.7571	1.2822
San Juan	-6.1677	2.3334	1.1734
Gila	-7.1863	2.5289	1.2688

The model for the EB estimate is as follows:

$$\ln V_{t,bf} = \bar{B}_0 + \bar{B}_1 \ln D + \bar{B}_2 \ln H$$

Table 27
Parameters of the EB Estimate for the Total
Scribner Board Foot Volume Equation

Douglas-fir			
National Forest	$\bar{B}_0$	$\bar{B}_1$	$\bar{B}_2$
A/S	-8.7172	1.7310	2.1386
Santa Fe	-5.8988	1.9393	1.3603
Kaibab	-7.2383	1.9186	1.6715
Lincoln	-9.8833	2.2214	2.0175
Dixie	-7.7519	2.0280	1.7199
Carson	-7.2533	1.9337	1.6777
Rio Grande	-6.0467	2.1218	1.3021

The model for the EB estimate is as follows:

$$\ln V_{t,bf} = \bar{B}_0 + \bar{B}_1 \ln D + \bar{B}_2 \ln H$$

Table 28
Parameters of the EB Estimate for the Total
Scribner Board Foot Volume Equation

White fir			
National Forest	$\tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$
A/S	-6.1484	2.1176	1.2978
Santa Fe	-6.5010	2.1349	1.3596
Kaibab	-6.8069	2.1679	1.4218
Lincoln	-6.1417	2.2265	1.1709
Dixie	-7.1584	2.1992	1.4704
Carson	-5.4070	2.0701	1.1570
Rio Grande	-6.2061	1.9884	1.4488

The model for the EB estimate is as follows:

$$\ln V_{t,bf} = \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln H$$

Table 29
Total Scribner Board Foot Volume
($V_{t.bf}$) Equation

Ponderosa Pine

National Forest	Equation
Tonto	$V_{t.bf} = 0.0005236 D^{2.9552} H^{1.0737}$
Apache-Sitgreaves	$V_{t.bf} = 0.0001533 D^{2.7756} H^{1.4645}$
Santa Fe	$V_{t.bf} = 0.001004 D^{2.4336} H^{1.2850}$
Prescott	$V_{t.bf} = 0.001415 D^{2.7850} H^{0.9138}$
Coconino	$V_{t.bf} = 0.0002407 D^{3.5571} H^{0.8032}$
Kaibab	$V_{t.bf} = 0.0009219 D^{2.6819} H^{1.1056}$
Lincoln	$V_{t.bf} = 0.001174 D^{2.3125} H^{1.3323}$
Dixie	$V_{t.bf} = 0.0008609 D^{2.5991} H^{1.1897}$
Carson	$V_{t.bf} = 0.001086 D^{2.8799} H^{0.9340}$
Rio Grande	$V_{t.bf} = 0.0004968 D^{2.7571} H^{1.2822}$
San Juan	$V_{t.bf} = 0.002096 D^{2.3334} H^{1.1734}$
Gila	$V_{t.bf} = 0.0007569 D^{2.5289} H^{1.2688}$

Table 30
Total Scribner Board Foot Volume
($V_{t.bf}$) Equation

Douglas-fir

National Forest	Equation
Apache-Sitgreaves	$V_{t.bf} = 0.0001637 D^{1.7310} H^{2.1386}$
Santa Fe	$V_{t.bf} = 0.0027426 D^{1.9393} H^{1.3603}$
Kaibab	$V_{t.bf} = 0.0007186 D^{1.9186} H^{1.6715}$
Lincoln	$V_{t.bf} = 0.0000510 D^{2.2214} H^{2.0175}$
Dixie	$V_{t.bf} = 0.0004299 D^{2.0280} H^{1.7199}$
Carson	$V_{t.bf} = 0.0007078 D^{1.9337} H^{1.6777}$
Rio Grande	$V_{t.bf} = 0.0023658 D^{2.1218} H^{1.3021}$

Table 31
Total Scribner Board Foot Volume
($V_{t.bf}$) Equation

White fir

National Forest	Equation
Apache-Sitgreaves	$V_{t.bf} = 0.0021368 D^{2.1176} H^{1.2978}$
Santa Fe	$V_{t.bf} = 0.0015020 D^{2.1349} H^{1.3596}$
Kaibab	$V_{t.bf} = 0.0011061 D^{2.1679} H^{1.4218}$
Lincoln	$V_{t.bf} = 0.0021514 D^{2.2265} H^{1.1709}$
Dixie	$V_{t.bf} = 0.0007783 D^{2.1992} H^{1.4704}$
Carson	$V_{t.bf} = 0.0044849 D^{2.0701} H^{1.1570}$
Rio Grande	$V_{t.bf} = 0.0020170 D^{1.9884} H^{1.4488}$

upper-stem diameter equals zero or when merchantable height equals total height. The ponderosa pine models expressed in a non-linear form are:

$$V_{m.ib} = V_{t.ib} - b_0 V_{t.ib}^{b_1} \frac{d_{ib}^{b_2}}{D^{b_3}} \quad (18)$$

and

$$V_{m.ib} = V_{t.ib} - V_{(t.ib)}^{b_1} \frac{(H-h)^{b_2}}{H^{b_3}} \quad (19)$$

where

$V_{t.ib}$ = total inside bark cubic foot volume

$V_{m.ib}$ = merchantable inside bark cubic foot volume

d_{ib} = inside bark upper-stem diameter in inches

D = dbh in inches

h = merchantable height in feet corresponding to d_{ib}

H = total tree height in feet

b_i = regression coefficients to be estimated.

Models (18) and (19) can be linearized with algebraic manipulation and a natural logarithmic transformation and expressed in a general linear form:

$$\ln(V_{t.ib} - V_{m.ib}) = \ln b_0 + b_1 \ln V_{t.ib} + b_2 \ln d_{ib} + b_3 \ln D \quad (20)$$

$$\ln(V_{t.ib} - V_{m.ib}) = b_1 \ln V_{t.ib} + b_2 \ln(H-h) + b_3 \ln H \quad (21)$$

Neither equation (18) nor (19) proved satisfactory for modeling merchantable cubic foot volume of Douglas-fir and white fir. The prediction of merchantable cubic foot volume to any upper-stem diameter was accomplished for Douglas-fir and white fir using the Burkhardt (1977) model:

$$V_{m.ib} = V_{t.ib} - b_0 V_{t.ib} \frac{d^{b_2}}{D^{b_3}} \quad (22)$$

to any upper-stem diameter was accomplished for Douglas-fir and white fir using the Burkhart (1977) model:

$$V_{m.ib} = V_{t.ib} - b_0 V_{t.ib} \frac{d^{b_2}}{D^{b_3}} \quad (22)$$

The prediction of merchantable volume to any merchantable height utilized a modeling procedure that is similar to the Newnham (1988) method for constructing a variable-form taper function:

$$V_{m.ib} = V_{t.ib} - V_{t.ib}^{a_1} \left[\frac{(H-h)}{H} \right]^{a_2 + a_3 \frac{(H-h)}{H}} \quad (23)$$

where

a_i = regression coefficients to be estimated.

Models (22) and (23) can be linearized with algebraic manipulation and a natural logarithmic transformation and expressed in a general linear form:

$$\ln(1 - V_{m.ib}/V_{t.ib}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D \quad (24)$$

$$\begin{aligned} \ln(V_{t.ib} - V_{m.ib}) = & a_1 \ln V_{t.ib} + a_2 \ln[(H-h)/H] \\ & + a_3[(H-h)/H] \ln[(H-h)/H] \end{aligned} \quad (25)$$

Models 20 and 21 were fitted to each observed bolt cut on sample trees from each national forest using ordinary least squares regression for ponderosa pine. Models 24 and 25 were

Table 32
Coefficients for the Merchantable Inside Bark Cubic
Foot Volume Equation (for any upper-stem diameter)
from OLS Regression by National Forest

Ponderosa Pine

National Forest	$\ln b_0$	b_1	b_2	b_3	Root MSE	R^2	No. of obs. (N)
Tonto	-2.1637	0.1738	3.3751	-1.4696	0.3061	0.9853	850
A/S	-0.7073	0.7715	3.2781	-2.7080	0.4106	0.9757	6843
S. Fe	-1.3126	0.5717	3.2698	-2.2356	0.4340	0.9743	2281
Prs.	-2.3814	0.2590	3.2293	-1.3818	0.2715	0.9894	2219
Coc.	-1.8675	0.4420	3.6052	-2.1596	0.3090	0.9844	2693
Kai.	-2.1846	0.3225	3.3398	-1.6376	0.3309	0.9838	2376
Lin.	-1.8028	0.5743	3.3192	-2.1133	0.2769	0.9879	2169
Dixie	-1.9363	0.1749	3.7628	-1.9202	0.2609	0.9761	109
Car.	-1.1335	0.5977	3.2906	-2.3130	0.3120	0.9847	1533
Rio.	-1.9681	0.2098	3.1530	-1.6821	0.2192	0.9833	40
S. J.	-1.9351	0.6216	3.2945	-2.1929	0.2936	0.9823	465
Gila	-2.6881	0.3929	3.5961	-1.8543	0.2961	0.9746	215

Note: obs.= observations; S.J. = San Juan
 A/S = Apache-Sitgreaves; S. Fe = Santa Fe
 Prs. = Prescott; Coc.= Coconino; Kai. = Kaibab
 Lin. = Lincoln; Car.= Carson; Rio. = Rio Grande

The least squares linear regression model for merchantable volume to any upper-stem diameter is as follows:

$$\ln(V_{t.ib} - V_{m.ib}) = \ln b_0 + b_1 \ln V_{t.ib} + b_2 \ln d_{ib} + b_3 \ln D$$

Table 33
Coefficients for the Merchantable Inside Bark Cubic Foot
Volume Equation (for any merchantable height)
from OLS Regression by National Forest

Ponderosa Pine

National Forest	b_1	b_2	b_3	Root MSE	R^2	Number of obs. (N)
Tonto	0.9459	2.5697	-2.4950	0.3065	0.9853	849
A/S	1.0036	2.6781	-2.6412	0.4022	0.9767	6843
Santa Fe	0.9331	2.7053	-2.5752	0.5208	0.9630	2281
Prescott	0.9441	2.7406	-2.6535	0.2910	0.9879	2219
Coconino	0.9739	2.3149	-2.2809	0.2881	0.9865	2693
Kaibab	0.8998	2.6519	-2.5145	0.3603	0.9808	2376
Lincoln	0.9315	2.6582	-2.5535	0.2718	0.9884	2169
Dixie	1.0202	2.4541	-2.4453	0.1849	0.9882	109
Carson	0.9378	2.7469	-2.6424	0.2800	0.9877	1533
Rio Grande	0.9854	2.7727	-2.7412	0.1166	0.9955	40
San Juan	0.9496	2.5965	-2.5376	0.2730	0.9848	465
Gila	0.9293	2.4718	-2.3960	0.2837	0.9769	215

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable volume to any merchantable height is as follows:

$$\ln(V_{t,ib} - V_{m,ib}) = b_1 \ln V_{t,ib} + b_2 \ln(H-h) + b_3 \ln H$$

fitted to each observed bolt cut for Douglas-fir and white fir. The estimated ponderosa pine coefficients for model (20) are listed in Table 32, and the estimated ponderosa pine coefficients for model (21) are listed in Table 33. The estimated Douglas-fir and white fir coefficients for model (24) are listed in Tables 34 and 35 respectively. The estimated Douglas-fir and white fir coefficients for model (25) are listed in Tables 36 and 37 respectively.

Table 34
Coefficients for the Merchantable Inside Bark Cubic Foot
Volume Equation (for any upper-stem diameter)
from OLS Regression by National Forest

Douglas-fir						
National Forest	$\ln b_0$	b_2	b_3	Root MSE	R^2	Number of obs. (N)
A/S	-0.4064	3.2591	-3.0143	0.2239	0.9887	266
Santa Fe	-0.2513	3.6405	-3.3986	0.3088	0.9490	648
Kaibab	-1.2026	3.4252	-2.9239	0.2446	0.9842	273
Lincoln	-0.2485	3.3353	-3.1285	0.2727	0.9786	1264
Dixie	-1.3627	3.4183	-2.9528	0.2090	0.9683	51
Carson	-1.4400	3.3400	-2.7250	0.2718	0.9856	147
Rio Grande	-1.9800	3.2929	-2.7502	0.2742	0.9705	213

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable volume to any upper-stem diameter is as follows:

$$\ln(1 - V_{m,ib}/V_{t,ib}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D$$

Table 35
Coefficients for the Merchantable Inside Bark Cubic Foot
Volume Equation (for any upper-stem diameter)
from OLS Regression by National Forest

White fir

National Forest	$\ln b_0$	b_2	b_3	Root MSE	R^2	Number of obs. (N)
A/S	-0.8424	3.3561	-2.9743	0.2198	0.9875	217
Santa Fe	-0.0853	3.3696	-3.2362	0.2725	0.9846	124
Kaibab	-1.1651	3.5033	-3.0388	0.2636	0.9823	220
Lincoln	-0.4938	3.4500	-3.1172	0.2836	0.9670	1970
Dixie	-0.8252	3.3456	-3.0893	0.2023	0.9732	59
Carson	-1.8915	3.4055	-2.6437	0.3503	0.9755	102
Rio Grande	0.0182	3.2614	-3.4357	0.2294	0.9744	18

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable volume to any upper-stem diameter is as follows:

$$\ln(1 - V_{m.ib}/V_{t.ib}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D$$

Table 36
Coefficients for the Merchantable Inside Bark Cubic Foot
Volume Equation (for any merchantable height)
from OLS Regression by National Forest

Douglas-fir

National Forest	a_1	a_2	a_3	Root MSE	R^2	Number of obs. (N)
A/S	1.0012	2.7728	-0.7757	0.2024	0.9946	266
Santa Fe	0.9762	2.5816	-1.1873	0.1759	0.9969	649
Kaibab	0.9709	2.6535	-1.1641	0.1829	0.9955	274
Lincoln	1.0004	2.6921	-0.7045	0.2239	0.9932	1265
Dixie	1.0689	2.5981	0.3764	0.1868	0.9898	52
Carson	0.9771	2.7056	-1.1975	0.2098	0.9939	148
Rio Grande	0.9863	2.7333	-0.7695	0.2407	0.9838	214

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable volume to any merchantable height is as follows:

$$\ln(V_{t,ib} - V_{m,ib}) = a_1 \ln V_{t,ib} + a_2 \ln[(H-h)/H] \\ + a_3[(H-h)/H] \ln[(H-h)/H]$$

Table 37
Coefficients for the Merchantable Inside Bark Cubic Foot
Volume Equation (for any merchantable height)
from OLS Regression by National Forest

White fir

National Forest	a_1	a_2	a_3	Root MSE	R^2	Number of obs. (N)
A/S	0.9747	2.6508	-0.8809	0.1966	0.9940	218
Santa Fe	0.9557	2.6361	-0.6758	0.2001	0.9951	125
Kaibab	1.0055	2.5293	-0.3501	0.1776	0.9957	221
Lincoln	0.9911	2.6112	-0.5847	0.1960	0.9935	1971
Dixie	1.0747	2.6914	0.2827	0.2071	0.9893	60
Carson	0.9553	2.5265	-1.1682	0.3382	0.9853	103
Rio Grande	1.4054	2.9041	3.5477	0.1752	0.9867	19

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable volume to any merchantable height is as follows:

$$\ln(V_{t.ib} - V_{m.ib}) = a_1 \ln V_{t.ib} + a_2 \ln[(H-h)/H] \\ + a_3[(H-h)/H] \ln[(H-h)/H]$$

Modeling Hyperparameters The covariate selected for the hyperparameter model was a unit scaler, which has the value of one. Therefore, each b_i or a_i was shrunk to its group mean. The model was estimated with the seemingly unrelated regression procedure:

$$\begin{aligned}
 b_i' &= Z \Gamma + \Delta_i \\
 \text{or } a_i' &= Z \Gamma + \Delta_i
 \end{aligned}
 \tag{26}$$

where

b_i' or a_i' = a (1 x p) vector of regression coefficients
from OLS estimation for national forest i

Z = an unit scaler

Γ = a (1 x p) vector of coefficients of
hyperparameters.

Δ_i = a (1 x p) error vector

$p = 4$, when the estimated hyperparameters are based
on the coefficients of the first-stage model
(20)

$p = 3$, when the estimated hyperparameters are based
on the coefficients of the first-stage models
(21), (24), and (25).

The estimated ponderosa pine hyperparameters based on the coefficients of the first-stage model (20) are listed in Table 38 and the ponderosa pine hyperparameters based on the coefficients of the first-stage model (21) are listed in Table 39. The estimated Douglas-fir and white fir hyperparameters based on the coefficients of the first-stage model (24) are listed in Tables 40 and 41 respectively, while the Douglas-fir and white fir hyperparameters based on the coefficients of the first-stage model (25) are listed in Tables 42 and 43 respectively.

EB Estimate The EB estimate of merchantable cubic foot volume followed the procedure used in the estimation of total inside bark cubic foot volume. The parameters of the EB estimate for the model based on upper-stem diameter are listed in Tables 44-46 and those for the model based on merchantable height are listed in Tables 47-49. The non-linear form of the equations for merchantable inside bark cubic foot volume are listed in Tables 50-52.

Table 38
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (20) and Associated t-test

Ponderosa Pine

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
$\ln b_0$	Z	1	-1.8397	0.1591	-11.561	0.0001
b_1	Z	1	0.4264	0.0580	7.3570	0.0001
b_2	Z	1	3.3761	0.0523	64.539	0.0001
b_3	Z	1	-1.9723	0.1116	-17.670	0.0001

Note: Z = unit scaler

Table 39
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (21) and Associated t-test

Ponderosa Pine

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
b_1	Z	1	0.9545	0.0100	95.605	0.0001
b_2	Z	1	2.6134	0.0401	65.175	0.0001
b_3	Z	1	-2.5397	0.0364	-69.745	0.0001

Note: Z = unit scaler

Table 40
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (24) and Associated t-test

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
$\ln b_0$	Z	1	-0.9845	0.2583	-3.811	0.0089
b_2	Z	1	3.3873	0.0480	70.548	0.0001
b_3	Z	1	-2.9847	0.0873	-34.196	0.0001

Note: Z = unit scaler

Table 41
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (24) and Associated t-test

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
$\ln b_0$	Z	1	-0.7550	0.2482	-3.042	0.0227
b_2	Z	1	3.3845	0.0295	114.752	0.0001
b_3	Z	1	-3.0764	0.0920	-33.436	0.0001

Note: Z = unit scaler

Table 42
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (25) and Associated t-test

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
a_1	Z	1	0.9973	0.0127	78.294	0.0001
a_2	Z	1	2.6767	0.0264	101.400	0.0001
a_3	Z	1	-0.7746	0.2087	-3.710	0.0100

Note: Z = unit scaler

Table 43
Estimated Hyperparameter Coefficients Based on the
First-Stage Model (25) and Associated t-test

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $\geq t $
a_1	Z	1	1.0575	0.0597	17.722	0.0001
a_2	Z	1	2.6499	0.0483	54.910	0.0001
a_3	Z	1	0.0244	0.6119	0.040	0.9695

Note: Z = unit scaler

Table 44
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based
on Upper-stem Diameter

Ponderosa Pine

National Forest	$\ln \tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$
Tonto	-2.0785	0.2001	3.3751	-1.5349
A/S	-0.7274	0.7661	3.2781	-2.6924
Santa Fe	-1.3561	0.5624	3.2701	-2.2085
Prescott	-2.3294	0.2711	3.2296	-1.4155
Coconino	-1.8744	0.4403	3.6044	-2.1544
Kaibab	-2.1486	0.3316	3.3399	-1.6622
Lincoln	-1.7963	0.5744	3.3193	-2.1155
Dixie	-2.0876	0.1726	3.7214	-1.8331
Carson	-1.1962	0.5850	3.2908	-2.2751
Rio Grande	-2.1773	0.2098	3.1779	-1.6194
San Juan	-1.9197	0.6101	3.2945	-2.1813
Gila	-2.4072	0.4355	3.5836	-1.9911

Model for inside bark merchantable cubic foot volume:

$$\ln(V_{t,ib} - V_{m,ib}) = \ln \tilde{B}_0 + \tilde{B}_1 \ln V_{t,ib} + \tilde{B}_2 \ln d_{ib} + \tilde{B}_3 \ln D$$

Table 45
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based on
on Upper-stem Diameter

Douglas-fir

National Forest	$\ln \tilde{B}_0$	$\tilde{B}_2$	$\tilde{B}_3$
Apache-Sitgreaves	-0.4830	3.2626	-2.9911
Santa Fe	-0.2481	3.5635	-3.3372
Kaibab	-1.1429	3.4271	-2.9458
Lincoln	-0.2652	3.3315	-3.1203
Dixie	-1.2801	3.4145	-2.9772
Carson	-1.2435	3.3683	-2.8150
Rio Grande	-1.9618	3.3483	-2.7945

Model for merchantable inside bark cubic foot volume:

$$\ln(1 - V_{m,ib}/V_{t,ib}) = \ln \tilde{B}_0 + \tilde{B}_2 \ln d_{ib} + \tilde{B}_3 \ln D$$

Table 46
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based on
on Upper-stem Diameter

White fir

National Forest	$\ln \bar{B}_0$	$\bar{B}_2$	$\bar{B}_3$
Apache-Sitgreaves	-0.8331	3.3653	-2.9848
Santa Fe	-0.2037	3.3744	-3.2002
Kaibab	-1.1662	3.4765	-3.0173
Lincoln	-0.5005	3.4478	-3.1134
Dixie	-0.7640	3.3540	-3.1154
Carson	-1.6572	3.4230	-2.7361
Rio Grande	-0.0201	3.2894	-3.4357

Model for merchantable inside bark cubic foot volume:

$$\ln(1 - V_{m,ib}/V_{t,ib}) = \ln \bar{B}_0 + \bar{B}_2 \ln d_{ib} + \bar{B}_3 \ln D$$

Table 47
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based
on Merchantable Height

Ponderosa Pine

National Forest	$\bar{B}_1$	$\bar{B}_2$	$\bar{B}_3$
Tonto	0.9475	2.5705	-2.4971
Apache-Sitgreaves	1.0026	2.6779	-2.6400
Santa Fe	0.9369	2.7022	-2.5776
Prescott	0.9441	2.7395	-2.6526
Coconino	0.9739	2.3155	-2.2814
Kaibab	0.9034	2.6517	-2.5183
Lincoln	0.9331	2.6580	-2.5551
Dixie	1.0004	2.4522	-2.4246
Carson	0.9384	2.7463	-2.6425
Rio Grande	0.9764	2.7712	-2.7298
San Juan	0.9509	2.5996	-2.5406
Gila	0.9495	2.4817	-2.4207

Model for merchantable inside bark cubic foot volume:

$$\ln(V_{t.ib} - V_{m.ib}) = \bar{B}_1 \ln V_{t.ib} + \bar{B}_2 \ln(H-h) + \bar{B}_3 \ln H$$

Table 48
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based
on Merchantable Height

Douglas-fir

National Forest	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$
Apache-Sitgreaves	0.9985	2.7624	-0.7686
Santa Fe	0.9766	2.5833	-1.1830
Kaibab	0.9733	2.6579	-1.1509
Lincoln	1.0004	2.6923	-0.7054
Dixie	1.0470	2.6224	0.0385
Carson	0.9638	2.6485	-0.9985
Rio Grande	0.9954	2.7474	-0.7269

Model for merchantable inside bark cubic foot volume:

$$\ln(V_{t,ib} - V_{m,ib}) = \tilde{A}_1 \ln V_{t,ib} + \tilde{A}_2 \ln[(H-h)/H] \\ + \tilde{A}_3 [(H-h)/H] \ln[(H-h)/H]$$

Table 49
Parameters of the EB Estimate for Inside Bark
Merchantable Cubic Foot Volume Based
on Merchantable Height

White fir

National Forest	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$
Apache-Sitgreaves	0.9770	2.6325	-0.7612
Santa Fe	0.9997	2.6183	-0.5025
Kaibab	1.0058	2.5312	-0.3369
Lincoln	0.9914	2.6089	-0.5716
Dixie	1.0650	2.6596	0.2818
Carson	0.9309	2.5743	-1.3019
Rio Grande	1.3095	2.8310	2.8119

Model for merchantable inside bark cubic foot volume:

$$\ln(V_{t,ib} - V_{m,ib}) = \tilde{A}_1 \ln V_{t,ib} + \tilde{A}_2 \ln[(H-h)/H] \\ + \tilde{A}_3 [(H-h)/H] \ln[(H-h)/H]$$

Table 50
 Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

Ponderosa Pine

National Forest	Equation	
Tonto	(1)	$V_{m.ib} = V_{t.ib} - 0.1251(V_{t.ib})^{0.2001} \frac{(d_{ib})^{3.3751}}{D^{1.5349}}$
	(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9475} \frac{(H-h)^{2.5705}}{H^{2.4971}}$
A/S	(1)	$V_{m.ib} = V_{t.ib} - 0.4832(V_{t.ib})^{0.7661} \frac{(d_{ib})^{3.2781}}{D^{2.6924}}$
	(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0026} \frac{(H-h)^{2.6779}}{H^{2.6400}}$
S.Fe	(1)	$V_{m.ib} = V_{t.ib} - 0.2577(V_{t.ib})^{0.5624} \frac{(d_{ib})^{3.2701}}{D^{2.2085}}$
	(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9369} \frac{(H-h)^{2.7022}}{H^{2.5776}}$
Prs.	(1)	$V_{m.ib} = V_{t.ib} - 0.0974(V_{t.ib})^{0.2711} \frac{(d_{ib})^{3.2296}}{D^{1.4155}}$
	(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9441} \frac{(H-h)^{2.7395}}{H^{2.6526}}$
Coc.	(1)	$V_{m.ib} = V_{t.ib} - 0.1534(V_{t.ib})^{0.4403} \frac{(d_{ib})^{3.6044}}{D^{2.1544}}$
	(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9739} \frac{(H-h)^{2.3155}}{H^{2.2814}}$

Table 50, continued

National Forest	Equation
Kaibab (1)	$V_{m.ib} = V_{t.ib} - 0.1166 (V_{t.ib})^{0.3316} \frac{(d_{ib})^{3.3399}}{D^{1.6622}}$
(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9034} \frac{(H-h)^{2.6517}}{H^{2.5183}}$
Lin. (1)	$V_{m.ib} = V_{t.ib} - 0.1659 (V_{t.ib})^{0.5744} \frac{(d_{ib})^{3.3193}}{D^{2.1155}}$
(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9331} \frac{(H-h)^{2.6580}}{H^{2.5551}}$
Dixie (1)	$V_{m.ib} = V_{t.ib} - 0.1240 (V_{t.ib})^{0.1726} \frac{(d_{ib})^{3.7214}}{D^{1.8331}}$
(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0004} \frac{(H-h)^{2.4522}}{H^{2.4246}}$
Carson (1)	$V_{m.ib} = V_{t.ib} - 0.3023 (V_{t.ib})^{0.5850} \frac{(d_{ib})^{3.2908}}{D^{2.2751}}$
(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9384} \frac{(H-h)^{2.7463}}{H^{2.6425}}$
Rio. (1)	$V_{m.ib} = V_{t.ib} - 0.1133 (V_{t.ib})^{0.2098} \frac{(d_{ib})^{3.1779}}{D^{1.6194}}$
(2)	$V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9764} \frac{(H-h)^{2.7712}}{H^{2.7298}}$

Table 50, continued

National Forest	Equation
S.J.	(1) $V_{m.ib} = V_{t.ib} - 0.1467(V_{t.ib})^{0.6101} \frac{(d_{ib})^{3.2945}}{D^{2.1813}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9509} \frac{(H-h)^{2.5996}}{H^{2.5406}}$
Gila	(1) $V_{m.ib} = V_{t.ib} - 0.0901(V_{t.ib})^{0.4355} \frac{(d_{ib})^{3.5836}}{D^{1.9911}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9495} \frac{(H-h)^{2.4817}}{H^{2.4207}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Prs. = Prescott; Coc. = Coconino; Lin. = Lincoln
 Rio. = Rio Grande; S.J. = San Juan.

Table 51
 Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

Douglas-fir

National Forest	Equation
A/S	(1) $V_{m.ib} = V_{t.ib} - 0.6169(V_{t.ib}) \frac{(d_{ib})^{3.2626}}{D^{2.9911}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9985} \frac{[(H-h)/H]^{2.7624}}{[(H-h)/H]^{0.7686[(H-h)/H]}}$
S.Fe	(1) $V_{m.ib} = V_{t.ib} - 0.7803(V_{t.ib}) \frac{(d_{ib})^{3.5635}}{D^{3.3372}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9766} \frac{[(H-h)/H]^{2.5833}}{[(H-h)/H]^{1.1830[(H-h)/H]}}$
Kaibab	(1) $V_{m.ib} = V_{t.ib} - 0.3189(V_{t.ib}) \frac{(d_{ib})^{3.4271}}{D^{2.9458}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9733} \frac{[(H-h)/H]^{2.6579}}{[(H-h)/H]^{1.1509[(H-h)/H]}}$
Lin.	(1) $V_{m.ib} = V_{t.ib} - 0.7670(V_{t.ib}) \frac{(d_{ib})^{3.3315}}{D^{3.1203}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0004} \frac{[(H-h)/H]^{2.6923}}{[(H-h)/H]^{0.7054[(H-h)/H]}}$

Table 51, continued

National Forest	Equation
Dixie	(1) $V_{m.ib} = V_{t.ib} - 0.2780(V_{t.ib}) \frac{(d_{ib})^{3.4145}}{D^{2.9772}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0470} \frac{[(H-h)/H]^{2.6224}}{[(H-h)/H]^{-0.0385[(H-h)/H]}}$
Carson	(1) $V_{m.ib} = V_{t.ib} - 0.2884(V_{t.ib}) \frac{(d_{ib})^{3.3683}}{D^{2.8150}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9638} \frac{[(H-h)/H]^{2.6485}}{[(H-h)/H]^{0.9985[(H-h)/H]}}$
Rio.	(1) $V_{m.ib} = V_{t.ib} - 0.1406(V_{t.ib}) \frac{(d_{ib})^{3.3483}}{D^{2.7945}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9954} \frac{[(H-h)/H]^{2.7474}}{[(H-h)/H]^{0.7269[(H-h)/H]}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Lin.=Lincoln; Rio.= Rio Grande

Table 52
 Merchantable Inside Bark Cubic Foot Volume ($V_{m.ib}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

White fir

National Forest	Equation
A/S	$(1) \quad V_{m.ib} = V_{t.ib} - 0.4347 (V_{t.ib}) \frac{(d_{ib})^{3.3653}}{D^{2.9848}}$ $(2) \quad V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9770} \frac{[(H-h)/H]^{2.6325}}{[(H-h)/H]^{0.7612[(H-h)/H]}}$
S.Fe	$(1) \quad V_{m.ib} = V_{t.ib} - 0.8157 (V_{t.ib}) \frac{(d_{ib})^{3.3744}}{D^{3.2002}}$ $(2) \quad V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9997} \frac{[(H-h)/H]^{2.6183}}{[(H-h)/H]^{0.5025[(H-h)/H]}}$
Kaibab	$(1) \quad V_{m.ib} = V_{t.ib} - 0.3116 (V_{t.ib}) \frac{(d_{ib})^{3.4765}}{D^{3.0173}}$ $(2) \quad V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0058} \frac{[(H-h)/H]^{2.5312}}{[(H-h)/H]^{0.3369[(H-h)/H]}}$
Lin.	$(1) \quad V_{m.ib} = V_{t.ib} - 0.6062 (V_{t.ib}) \frac{(d_{ib})^{3.4478}}{D^{3.1134}}$ $(2) \quad V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9914} \frac{[(H-h)/H]^{2.6089}}{[(H-h)/H]^{0.5716[(H-h)/H]}}$

Table 52, continued

National Forest	Equation
Dixie	(1) $V_{m.ib} = V_{t.ib} - 0.4658 (V_{t.ib}) \frac{(d_{ib})^{3.3540}}{D^{3.1154}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.0650} \frac{[(H-h)/H]^{2.6596}}{[(H-h)/H]^{-0.2818[(H-h)/H]}}$
Carson	(1) $V_{m.ib} = V_{t.ib} - 0.1907 (V_{t.ib}) \frac{(d_{ib})^{3.4230}}{D^{2.7361}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{0.9309} \frac{[(H-h)/H]^{2.5743}}{[(H-h)/H]^{1.3019[(H-h)/H]}}$
Rio.	(1) $V_{m.ib} = V_{t.ib} - 0.9801 (V_{t.ib}) \frac{(d_{ib})^{3.2894}}{D^{3.4357}}$
	(2) $V_{m.ib} = V_{t.ib} - (V_{t.ib})^{1.3095} \frac{[(H-h)/H]^{2.8310}}{[(H-h)/H]^{-2.8119[(H-h)/H]}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Lin.=Lincoln; Rio.= Rio Grande

Merchantable Outside Bark Cubic
Foot Volume Equation

Since only a very small number of the ponderosa pine sample trees provided information of outside bark cubic foot volume, the model for merchantable outside bark volume was not constructed for a specific national forest. Rather, one equation was constructed for all national forests. The Douglas-fir and white fir data sets contained considerably more outside bark observations, however neither data set contained outside bark information for the Rio Grande National Forest. Therefore, we resolved to construct southwestern merchantable outside bark volume equations by species, where merchantable outside bark cubic foot volume is a function of merchantable inside bark cubic foot volume. The model has the following form:

$$V_{m.ob} = b_0 V_{m.ib}^{b_1} \quad (27)$$

where

$V_{m.ob}$ = merchantable outside bark cubic foot volume

$V_{m.ib}$ = merchantable inside bark cubic foot volume

b_i = regression coefficient.

The model (27) can be linearized by making a natural logarithm transformation:

$$\ln V_{m.ob} = \ln b_0 + b_1 \ln V_{m.ib} \quad (28)$$

Model (28) was fitted to a total of 1037 observed bolt cuts from ponderosa pine sample trees that possessed outside bark information by the least squares procedure. The bolt cut at the stump height was excluded since the observed merchantable volume is zero by definition. The linear form with estimated coefficients for ponderosa pine is expressed as:

$$\ln V_{m.ob} = 0.3800 + 0.9621 \ln V_{m.ib} \quad (29)$$

The equation may be expressed in non-linear form as:

Ponderosa Pine

$$V_{m.ob} = 1.4623 V_{m.ib}^{0.9621} \quad (30)$$

The coefficient of determination for equation (29) exceeds 0.995. Model (28) was fitted to a total of 2594 and 3089 observed Douglas-fir and white fir bolt cuts. The coefficient of determination for Douglas-fir exceeds 0.997 while the coefficient of determination for white fir exceeds 0.994. The merchantable outside bark equations for Douglas-fir and white fir may be expressed in non-linear form as:

Douglas-fir

$$V_{m.ob} = 1.3988 V_{m.ib}^{0.9875}$$

White fir

$$V_{m.ob} = 1.4462 V_{m.ib}^{0.9813}$$

Merchantable Scribner Board
Foot Volume Equations

Model Selection The following non-linear models were used to predict the merchantable Scribner board foot volume for ponderosa pine:

$$V_{m.bf} = V_{t.bf} - b_0 V_{t.bf}^{b_1} \frac{d_{ib}^{b_2}}{D^{b_3}} \quad (31)$$

and

$$V_{m.bf} = V_{t.bf} - V_{t.bf}^{b_1} \frac{(H-h)^{b_2}}{H^{b_3}} \quad (32)$$

where $V_{t.bf}$ = total Scribner board foot volume
 $V_{m.bf}$ = merchantable Scribner board foot volume.

Models (31) and (32) can be linearized with algebraic manipulation and a natural logarithmic transformation and expressed in a general linear form:

$$\begin{aligned} \ln(V_{t.bf} - V_{m.bf}) &= \ln b_0 + b_1 \ln V_{t.bf} \\ &+ b_2 \ln d_{ib} + b_3 \ln D \end{aligned} \quad (33)$$

and

$$\begin{aligned} \ln(V_{t.bf} - V_{m.bf}) &= b_1 \ln V_{t.bf} + b_2 \ln(H-h) \\ &+ b_3 \ln H \end{aligned} \quad (34)$$

Neither equation (31) nor (32) proved satisfactory for modeling merchantable board foot volume of Douglas-fir and white fir. The prediction of merchantable board foot volume to any upper-stem diameter was accomplished for Douglas-fir and white fir using the Burkhardt (1977) model:

$$V_{m.bf} = V_{t.bf} - b_0 V_{t.bf} \frac{d^{b_2}}{D^{b_3}} \quad (35)$$

The prediction of merchantable volume to any merchantable height utilized a modeling procedure that is similar to the Newnham (1988) method for constructing a variable-form taper function:

$$V_{m.bf} = V_{t.bf} - V_{t.bf}^{a_1} \left[\frac{(H-h)}{H} \right]^{a_2 + a_3 \frac{(H-h)}{H}} \quad (36)$$

where

a_i = regression coefficients to be estimated.

Models (35) and (36) can be linearized with algebraic manipulation and a natural logarithmic transformation and

expressed in a general linear form:

$$\ln(1 - V_{m,bf}/V_{t,bf}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D \quad (37)$$

$$\begin{aligned} \ln(V_{t,bf} - V_{m,bf}) = & a_1 \ln V_{t,bf} + a_2 \ln[(H-h)/H] \\ & + a_3[(H-h)/H] \ln[(H-h)/H] \end{aligned} \quad (38)$$

Models (33) and (34) were fitted to each observed bolt cut on ponderosa pine sample trees from each national forest using ordinary least squares. The estimated ponderosa pine regression coefficients for model (33) are listed in Table 53, and those for model (34) are listed in Table 54. Models (37) and (38) were fitted to each observed bolt cut on Douglas-fir and white fir sample trees by national forest using ordinary least squares. The estimated regression coefficients for model (37) are listed in Tables 55 and 56 for Douglas-fir and white fir respectively. The estimated regression coefficients for model (38) are listed in Tables 57 and 58 for Douglas-fir and white fir respectively.

Modeling Hyperparameters The ponderosa pine hyperparameter models for coefficients from models (33) and (34) and the Douglas-fir and white fir hyperparameter models for coefficients from models (37) and (38) were the same as those used in the estimation of merchantable inside bark cubic foot volume. The hyperparameters were estimated by the

Table 53

Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any upper-stem diameter)
from OLS Estimation by National Forest

Ponderosa Pine

National Forest	$\ln b_0$	b_1	b_2	b_3	Root MSE	R^2	Number of obs. (N)
Tonto	-2.6739	0.3428	6.1686	-3.7347	0.7019	0.9289	632
A/S	0.7432	1.1642	5.4523	-5.8692	0.7319	0.8984	4755
S.Fe	-1.6727	0.7425	5.9575	-4.7112	0.6498	0.9871	1825
Prs.	-3.7577	0.2701	5.5090	-2.6547	0.6994	0.9368	1716
Coc.	-2.9244	0.5180	5.1943	-3.1426	0.5011	0.9373	1862
Kai.	-2.1447	0.6053	5.6723	-3.9977	0.6747	0.9425	1874
Lin.	-3.1091	0.7368	5.6896	-3.9787	0.6375	0.9421	1717
Dixie	-3.3652	-0.4435	5.9938	-1.8824	0.5048	0.8885	65
Car.	-2.4937	0.6282	6.0164	-4.2069	0.6876	0.9340	1166
Rio.	-2.9891	0.5074	6.3086	-4.5264	0.3178	0.9734	18
S.J.	-3.6227	0.6008	5.7852	-3.7720	0.6466	0.9497	327
Gila	-1.1300	1.1066	7.3392	-6.9006	0.7301	0.9151	119

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
Prs. = Prescott; Coc. = Coconino; Kai. = Kaibab
Lin. = Lincoln; Car. = Carson; Rio. = Rio Grande
S.J. = San Juan obs. = observations.

The least squares regression model for merchantable Scribner board foot volume to any upper-stem diameter is as follows:

$$\ln(V_{t,bf} - V_{m,bf}) = \ln b_0 + b_1 \ln V_{t,bf} + b_2 \ln d_{ib} + b_3 \ln D$$

Table 54
Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any merchantable height)
from OLS Estimation by National Forest

Ponderosa Pine

National Forest	b_1	b_2	b_3	Root MSE	R^2	Number of obs. (N)
Tonto	0.9134	6.1495	-5.7506	0.7143	0.9266	632
A/S	1.0391	5.4518	-5.3766	0.7335	0.8980	4755
Santa Fe	1.0910	5.9462	-5.9482	0.6589	0.9457	1825
Prescott	1.0597	5.4854	-5.3848	0.7254	0.9321	1716
Coconino	0.9596	5.1687	-4.9349	0.5200	0.9326	1862
Kaibab	1.0599	5.6585	-5.5768	0.6875	0.9404	1874
Lincoln	1.3001	5.6540	-6.0974	0.6655	0.9369	1717
Dixie	0.3272	5.8795	-4.3916	0.5314	0.8803	65
Carson	1.0946	6.0038	-5.9311	0.6991	0.9319	1166
Rio Grande	1.4410	6.2575	-7.2900	0.3172	0.9766	18
San Juan	1.4624	5.7523	-6.5949	0.6753	0.9455	327
Gila	1.2975	7.3339	-7.6433	0.7296	0.9166	119

Note: A/S = Apache-Sitgreaves obs.= observations

The least squares regression model for merchantable Scribner board foot volume to any merchantable height is as follows:

$$\ln(V_{t,bf} - V_{m,bf}) = b_1 \ln V_{t,bf} + b_2 \ln(H-h) + b_3 \ln H$$

Table 55
Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any upper-stem diameter)
from OLS Estimation by National Forest

Douglas-fir

National Forest	$\ln b_0$	b_2	b_3	Root MSE	R^2	Number of obs. (N)
A/S	-1.9219	5.5544	-4.5756	0.4292	0.9197	205
Santa Fe	-0.9879	5.1061	-4.4923	0.4340	0.8808	560
Kaibab	-2.2360	5.9513	-4.8539	0.4365	0.9163	210
Lincoln	-1.4433	5.4127	-4.6035	0.5261	0.8927	887
Dixie	-3.2087	4.6537	-3.5732	0.3478	0.8921	31
Carson	-2.9685	6.4873	-5.0027	0.5635	0.8885	114
Rio Grande	-5.8180	5.8997	-4.1678	0.6599	0.8976	147

Note: A/S = Apache-Sitgreaves obs.= observations

The least squares regression model for merchantable Scribner board foot volume to any upper-stem diameter is as follows:

$$\ln(1 - V_{m.bf}/V_{t.bf}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D$$

Table 56
Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any upper-stem diameter)
from OLS Estimation by National Forest

White fir

National Forest	$\ln b_0$	b_2	b_3	Root MSE	R^2	Number of obs. (N)
A/S	-2.4096	5.9131	-4.7561	0.3964	0.9331	169
Santa Fe	-0.7674	6.1566	-5.5201	0.4660	0.9166	97
Kaibab	-1.7930	5.7756	-4.9082	0.3474	0.9444	168
Lincoln	-2.3409	5.4926	-4.3327	0.5668	0.8750	1556
Dixie	-2.5813	5.4867	-4.5212	0.4023	0.8887	39
Carson	-2.5382	5.7178	-4.5157	0.6812	0.8217	80
Rio Grande	-7.1342	6.7490	-4.1550	0.3371	0.9637	11

Note: A/S = Apache-Sitgreaves obs.= observations

The least squares regression model for merchantable Scribner board foot volume to any upper-stem diameter is as follows:

$$\ln(1 - V_{m.bf}/V_{t.bf}) = \ln b_0 + b_2 \ln d_{ib} + b_3 \ln D$$

Table 57
Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any merchantable height)
from OLS Regression by National Forest

Douglas-fir

National Forest	a_1	a_2	a_3	Root MSE	R^2	Number of obs. (N)
A/S	1.0591	4.0904	-0.4059	0.4449	0.9891	207
Santa Fe	0.9974	3.4806	-1.6577	0.2608	0.9972	561
Kaibab	0.9955	4.4130	-2.9026	0.3367	0.9939	211
Lincoln	1.0475	4.0531	-0.9822	0.4593	0.9889	889
Dixie	1.2135	3.4001	3.3275	0.2772	0.9935	32
Carson	0.9893	5.0913	-4.1166	0.4931	0.9854	115
Rio Grande	1.2398	4.7254	1.5177	0.5132	0.9708	148

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable Scribner board foot volume to any merchantable height is as follows:

$$\ln(V_{t.bf} - V_{m.bf}) = a_1 \ln V_{t.bf} + a_2 \ln[(H-h)/H] \\ + a_3[(H-h)/H] \ln[(H-h)/H]$$

Table 58
Coefficients for the Merchantable Scribner Board Foot
Volume Equation (for any merchantable height)
from OLS Regression by National Forest

White fir

National Forest	a_1	a_2	a_3	Root MSE	R^2	Number of obs. (N)
A/S	1.0380	4.5358	-1.6422	0.4019	0.9896	170
Santa Fe	0.9998	4.7096	-3.1344	0.3327	0.9942	98
Kaibab	1.0142	4.1360	-1.9197	0.2462	0.9967	169
Lincoln	1.0424	4.1500	-1.0048	0.4724	0.9855	1557
Dixie	1.3058	3.7713	4.5305	0.4119	0.9859	40
Carson	0.9801	4.4318	-3.3119	0.5955	0.9805	81
Rio Grande	1.8956	5.3469	10.1792	0.1721	0.9944	12

Note: obs. = observations A/S = Apache-Sitgreaves

The least squares linear regression model for merchantable Scribner board foot volume to any merchantable height is as follows:

$$\ln(V_{t,bf} - V_{m,bf}) = a_1 \ln V_{t,bf} + a_2 \ln[(H-h)/H] \\ + a_3 [(H-h)/H] \ln[(H-h)/H]$$

seemingly unrelated regression procedure, and the estimated parameters, based on the coefficients from models (33) and (34) and models (37) and (38) are listed in Tables 59-64.

EB Estimate The EB estimate of merchantable Scribner board foot volume followed the procedure used in the estimation of total inside bark cubic foot volume. The parameters of the EB estimate for the model based on upper-stem diameter are listed in Tables 65-67 and parameters for the model based on merchantable height are listed in Tables 68-70. The non-linear form of the equations for merchantable Scribner board foot volume are listed in Tables 71-73.

Table 59
Estimated Hyperparameter Coefficients Based on the
First-stage Model (33) and Associated t-test

Ponderosa Pine						
Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob > t
$\ln b_0$	Z	1	-2.4283	0.3647	-6.659	0.0001
b_1	Z	1	0.5649	0.1193	4.735	0.0001
b_2	Z	1	5.9239	0.1579	37.524	0.0001
b_3	Z	1	-4.1148	0.3866	-10.644	0.0001

Note: Z = unit scaler

Table 60
Estimated Hyperparameter Coefficients Based on the
First-stage Model (34) and Associated t-test

Ponderosa Pine

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
b_1	Z	1	1.0871	0.0863	12.595	0.0001
b_2	Z	1	5.8951	0.1583	37.242	0.0001
b_3	Z	1	-5.9100	0.2664	-22.185	0.0001

Note: Z = unit scaler

Table 61
Estimated Hyperparameter Coefficients Based on the
First-stage Model (37) and Associated t-test

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
$\ln b_0$	Z	1	-2.5638	0.5274	-4.861	0.0028
b_2	Z	1	5.5808	0.2275	24.526	0.0001
b_3	Z	1	-4.4670	0.1797	-24.859	0.0001

Note: Z = unit scaler

Table 62
Estimated Hyperparameter Coefficients Based on the
First-stage Model (37) and Associated t-test

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
$\ln b_0$	Z	1	-2.7949	0.7624	-3.666	0.0105
b_2	Z	1	5.8988	0.1671	35.307	0.0001
b_3	Z	1	-4.6727	0.1699	-27.510	0.0001

Note: Z = unit scaler

Table 63
Estimated Hyperparameter Coefficients Based on the
First-stage Model (38) and Associated t-test

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
a_1	Z	1	1.0774	0.0399	26.978	0.0001
a_2	Z	1	4.1791	0.2342	17.844	0.0001
a_3	Z	1	-0.7457	0.9603	-0.777	0.4669

Note: Z = unit scaler

Table 64
Estimated Hyperparameter Coefficients Based on the
First-stage Model (38) and Associated t-test

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
a_1	Z	1	1.1823	0.1260	9.382	0.0001
a_2	Z	1	4.4402	0.1907	23.279	0.0001
a_3	Z	1	0.5281	1.8922	0.279	0.7895

Note: Z = unit scaler

Table 65
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Upper-stem Diameter

Ponderosa Pine				
National Forest	$\ln \bar{B}_0$	$\bar{B}_1$	$\bar{B}_2$	$\bar{B}_3$
Tonto	-2.5607	0.3729	6.1574	-3.8237
Apache-Sitgreaves	0.6864	1.1549	5.4551	-5.8344
Santa Fe	-1.6953	0.7379	5.9575	-4.6945
Prescott	-3.6742	0.2873	5.5090	-2.7152
Coconino	-2.9102	0.5192	5.1973	-3.1521
Kaibab	-2.1564	0.6031	5.6730	-3.9902
Lincoln	-3.0689	0.7383	5.6894	-3.9945
Dixie	-2.9298	-0.1421	5.8639	-2.5084
Carson	-2.4533	0.6343	6.0126	-4.2292
Rio Grande	-2.9899	0.6230	6.3846	-4.7889
San Juan	-3.4141	0.6202	5.7902	-3.8800
Gila	-1.3972	1.0291	7.1264	-6.4910

Model for merchantable Scribner board foot volume based on upper-stem diameter:

$$\ln(V_{t.bf} - V_{m.bf}) = \ln \bar{B}_0 + \bar{B}_1 \ln V_{t.bf} + \bar{B}_2 \ln d_{ib} + \bar{B}_3 \ln D$$

Table 66
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Upper-stem Diameter

Douglas-fir

National Forest	$\ln \bar{B}_0$	$\bar{B}_2$	$\bar{B}_3$
Apache-Sitgreaves	-1.8974	5.5583	-4.5877
Santa Fe	-1.0840	5.1079	-4.4631
Kaibab	-2.3384	5.9256	-4.7970
Lincoln	-1.4672	5.4124	-4.5956
Dixie	-2.8496	4.8532	-3.8523
Carson	-2.6331	6.3990	-5.0489
Rio Grande	-6.6257	5.9100	-3.8913

Model for merchantable Scribner board foot volume based on upper-stem diameter:

$$\ln(1 - V_{m.bf}/V_{t.bf}) = \ln \bar{B}_0 + \bar{B}_2 \ln d_{ib} + \bar{B}_3 \ln D$$

Table 67
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Upper-stem Diameter

White fir

National Forest	$\ln \tilde{B}_0$	$\tilde{B}_2$	$\tilde{B}_3$
Apache-Sitgreaves	-2.3990	5.9002	-4.7496
Santa Fe	-1.0569	6.0253	-5.3111
Kaibab	-1.8312	5.7769	-4.8956
Lincoln	-2.3216	5.4969	-4.3430
Dixie	-2.8697	5.7620	-4.6412
Carson	-2.3673	5.7415	-4.5981
Rio Grande	-6.3845	6.5905	-4.2893

Model for merchantable Scribner board foot volume based on upper-stem diameter:

$$\ln(1 - V_{m,bf}/V_{t,bf}) = \ln \tilde{B}_0 + \tilde{B}_2 \ln d_{ib} + \tilde{B}_3 \ln D$$

Table 68
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Merchantable Height

Ponderosa Pine			
National Forest	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$
Tonto	0.9197	6.1412	-5.7564
Apache-Sitgreaves	1.0075	5.3622	-5.5062
Santa Fe	1.0910	5.9459	-5.9479
Prescott	1.0584	5.4862	-5.3829
Coconino	0.9596	5.1719	-4.9375
Kaibab	1.0595	5.6588	-5.5763
Lincoln	1.2949	5.6549	-6.0875
Dixie	0.5552	5.7887	-4.7640
Carson	1.0929	5.9999	-5.9248
Rio Grande	1.5377	6.3459	-7.5285
San Juan	1.4215	5.7597	-6.5203
Gila	1.2111	7.0984	-7.4199

Model for merchantable Scribner board foot volume based on merchantable height:

$$\ln(V_{t.bf} - V_{m.bf}) = \tilde{B}_1 \ln V_{t.bf} + \tilde{B}_2 \ln(H-h) + \tilde{B}_3 \ln H$$

Table 69
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Merchantable Height

Douglas-fir			
National Forest	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$
Apache-Sitgreaves	1.0588	4.1486	-0.6025
Santa Fe	0.9978	3.4781	-1.6383
Kaibab	0.9968	4.3996	-2.8410
Lincoln	1.0476	4.0581	-0.9959
Dixie	1.1866	3.4844	2.6618
Carson	0.9982	4.9791	-3.6974
Rio Grande	1.2156	4.6863	1.2289

Model for merchantable Scribner board foot volume based on merchantable height:

$$\ln(V_{t.bf} - V_{m.bf}) = \tilde{A}_1 \ln V_{t.bf} + \tilde{A}_2 \ln[(H-h)/H] + \tilde{A}_3 [(H-h)/H] \ln[(H-h)/H]$$

Table 70
Parameters of the EB Estimate for Merchantable Scribner
Board Foot Volume Based on Merchantable Height

White fir

National Forest	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$
Apache-Sitgreaves	1.0406	4.4689	-1.4990
Santa Fe	1.0030	4.6475	-2.9466
Kaibab	1.0143	4.1473	-1.9295
Lincoln	1.0422	4.1519	-1.0124
Dixie	1.2678	4.0344	3.1949
Carson	0.9907	4.4024	-2.9078
Rio Grande	1.8420	5.2867	9.5527

Model for merchantable Scribner board foot volume based on merchantable height:

$$\ln(V_{t,bf} - V_{m,bf}) = \tilde{A}_1 \ln V_{t,bf} + \tilde{A}_2 \ln[(H-h)/H] + \tilde{A}_3[(H-h)/H] \ln[(H-h)/H]$$

Table 71
 Merchantable Scribner Board Foot Volume ($V_{m.bf}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

Ponderosa Pine

National Forest	Equation
Tonto	(1) $V_{m.bf} = V_{t.bf} - 0.0773 (V_{t.bf})^{0.3729} \frac{(d_{ib})^{6.1574}}{D^{3.8237}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9197} \frac{(H-h)^{6.1412}}{H^{5.7564}}$
A/S	(1) $V_{m.bf} = V_{t.bf} - 1.9866 (V_{t.bf})^{1.1549} \frac{(d_{ib})^{5.4551}}{D^{5.8344}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0075} \frac{(H-h)^{5.3622}}{H^{5.5062}}$
S.Fe	(1) $V_{m.bf} = V_{t.bf} - 0.1835 (V_{t.bf})^{0.7379} \frac{(d_{ib})^{5.9575}}{D^{4.6945}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0910} \frac{(H-h)^{5.9459}}{H^{5.9479}}$
Prs.	(1) $V_{m.bf} = V_{t.bf} - 0.0254 (V_{t.bf})^{0.2873} \frac{(d_{ib})^{5.5090}}{D^{2.7152}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0584} \frac{(H-h)^{5.4862}}{H^{5.3829}}$
Coc.	(1) $V_{m.bf} = V_{t.bf} - 0.0545 (V_{t.bf})^{0.5192} \frac{(d_{ib})^{5.1973}}{D^{3.1521}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9596} \frac{(H-h)^{5.1719}}{H^{4.9375}}$

Table 71, continued

National Forest	Equation
Kaibab (1)	$V_{m.bf} = V_{t.bf} - 0.1157 (V_{t.bf})^{0.6031} \frac{(d_{ib})^{5.6730}}{D^{3.9902}}$
(2)	$V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0595} \frac{(H-h)^{5.6588}}{H^{5.5763}}$
Lin. (1)	$V_{m.bf} = V_{t.bf} - 0.0465 (V_{t.bf})^{0.7383} \frac{(d_{ib})^{5.6894}}{D^{3.9945}}$
(2)	$V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.2949} \frac{(H-h)^{5.6549}}{H^{6.0875}}$
Dixie (1)	$V_{m.bf} = V_{m.bf} - 0.0534 (V_{t.bf})^{-0.1421} \frac{(d_{ib})^{5.8639}}{D^{2.5084}}$
(2)	$V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.5552} \frac{(H-h)^{5.7887}}{H^{4.7640}}$
Carson (1)	$V_{m.bf} = V_{t.bf} - 0.0860 (V_{t.bf})^{0.6343} \frac{(d_{ib})^{6.0126}}{D^{4.2292}}$
(2)	$V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0929} \frac{(H-h)^{5.9999}}{H^{5.9248}}$
Rio. (1)	$V_{m.bf} = V_{t.bf} - 0.0503 (V_{t.bf})^{0.6230} \frac{(d_{ib})^{6.3846}}{D^{4.7889}}$
(2)	$V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.5377} \frac{(H-h)^{6.3459}}{H^{7.5285}}$

Table 71, continued

National Forest	Equation
S.J.	(1) $V_{m.bf} = V_{t.bf} - 0.0329 (V_{t.bf})^{0.6202} \frac{(d_{ib})^{5.7902}}{D^{3.8800}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.4215} \frac{(H-h)^{5.7597}}{H^{6.5203}}$
Gila	(1) $V_{m.bf} = V_{t.bf} - 0.2473 (V_{t.bf})^{1.0291} \frac{(d_{ib})^{7.1264}}{D^{6.4910}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.2111} \frac{(H-h)^{7.0984}}{H^{7.4199}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Prs. = Prescott; Coc. = Coconino; Lin. = Lincoln
 Rio. = Rio Grande; S.J. = San Juan.

Table 72
 Merchantable Scribner Board Foot Volume ($V_{m.bf}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

Douglas-fir

National Forest	Equation
A/S	(1) $V_{m.bf} = V_{t.bf} - 0.1500(V_{t.bf}) \frac{(d_{ib})^{5.5583}}{D^{4.5877}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0588} \frac{[(H-h)/H]^{4.1486}}{[(H-h)/H]^{0.0625[(H-h)/H]}}$
S.Fe	(1) $V_{m.bf} = V_{t.bf} - 0.3383(V_{t.bf}) \frac{(d_{ib})^{5.1079}}{D^{4.4631}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9978} \frac{[(H-h)/H]^{3.4781}}{[(H-h)/H]^{1.6383[(H-h)/H]}}$
Kaibab	(1) $V_{m.bf} = V_{t.bf} - 0.0965(V_{t.bf}) \frac{(d_{ib})^{5.9256}}{D^{4.7970}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9968} \frac{[(H-h)/H]^{4.3996}}{[(H-h)/H]^{2.8410[(H-h)/H]}}$
Lin.	(1) $V_{m.bf} = V_{t.bf} - 0.2306(V_{t.bf}) \frac{(d_{ib})^{5.4124}}{D^{4.5956}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0476} \frac{[(H-h)/H]^{4.0581}}{[(H-h)/H]^{0.9959[(H-h)/H]}}$

Table 72, continued

National Forest	Equation
Dixie	(1) $V_{m.bf} = V_{t.bf} - 0.0579 (V_{t.bf}) \frac{(d_{ib})^{4.8532}}{D^{3.8523}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.1866} \frac{[(H-h)/H]^{3.4844}}{[(H-h)/H]^{-2.6618[(H-h)/H]}}$
Carson	(1) $V_{m.bf} = V_{t.bf} - 0.0719 (V_{t.bf}) \frac{(d_{ib})^{6.3990}}{D^{5.0489}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9982} \frac{[(H-h)/H]^{4.9791}}{[(H-h)/H]^{3.6974[(H-h)/H]}}$
Rio.	(1) $V_{m.bf} = V_{t.bf} - 0.0013 (V_{t.bf}) \frac{(d_{ib})^{5.9100}}{D^{3.8913}}$
	(2) $V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.2156} \frac{[(H-h)/H]^{4.6863}}{[(H-h)/H]^{-1.2289[(H-h)/H]}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Lin.=Lincoln; Rio.= Rio Grande

Table 73
 Merchantable Scribner Board Foot Volume ($V_{m.bf}$)
 Equations Based on Upper-stem Diameter (1)
 and on Merchantable Height (2)

White fir

National Forest	Equation
A/S	$(1) \quad V_{m.bf} = V_{t.bf} - 0.0908 (V_{t.bf}) \frac{(d_{ib})^{5.9002}}{D^{4.7496}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0406} \frac{[(H-h)/H]^{4.4689}}{[(H-h)/H]^{1.4990[(H-h)/H]}}$
S.Fe	$(1) \quad V_{m.bf} = V_{t.bf} - 0.3475 (V_{t.bf}) \frac{(d_{ib})^{6.0253}}{D^{5.3111}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0030} \frac{[(H-h)/H]^{4.6475}}{[(H-h)/H]^{2.9466[(H-h)/H]}}$
Kaibab	$(1) \quad V_{m.bf} = V_{t.bf} - 0.1602 (V_{t.bf}) \frac{(d_{ib})^{5.7769}}{D^{4.8956}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0143} \frac{[(H-h)/H]^{4.1473}}{[(H-h)/H]^{1.9295[(H-h)/H]}}$
Lin.	$(1) \quad V_{m.bf} = V_{t.bf} - 0.0981 (V_{t.bf}) \frac{(d_{ib})^{5.4969}}{D^{4.3430}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.0422} \frac{[(H-h)/H]^{4.1519}}{[(H-h)/H]^{1.0124[(H-h)/H]}}$

Table 73, continued

National Forest	Equation
Dixie	$(1) \quad V_{m.bf} = V_{t.bf} - 0.0567(V_{t.bf}) \frac{(d_{ib})^{5.7620}}{D^{4.6412}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.2678} \frac{[(H-h)/H]^{4.0344}}{[(H-h)/H]^{-3.1949[(H-h)/H]}}$
Carson	$(1) \quad V_{m.bf} = V_{t.bf} - 0.0937(V_{t.bf}) \frac{(d_{ib})^{5.7415}}{D^{4.5981}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{0.9907} \frac{[(H-h)/H]^{4.4024}}{[(H-h)/H]^{2.9078[(H-h)/H]}}$
Rio.	$(1) \quad V_{m.bf} = V_{t.bf} - 0.0017(V_{t.bf}) \frac{(d_{ib})^{6.5905}}{D^{4.2893}}$ $(2) \quad V_{m.bf} = V_{t.bf} - (V_{t.bf})^{1.8420} \frac{[(H-h)/H]^{5.2867}}{[(H-h)/H]^{-9.5527[(H-h)/H]}}$

Note: A/S = Apache-Sitgreaves; S.Fe = Santa Fe
 Lin.=Lincoln; Rio.= Rio Grande

Inside Bark Taper Function

Model Selection Newnham (1988) and Kozak (1988)

introduced a "Variable-form" taper function which describes

tree taper with a continuous function using a changing exponent to compensate for the form changes of different tree sections.

The variable-form taper model proposed by Kozak (1988) has the following form:

$$d = b_0 D^{b_1} b_2^D X^c \quad (39)$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I})$$

$$C = b_3 Z^2 + b_4 \ln(Z+0.001) + b_5 \sqrt{Z} + b_6 e^Z + b_7 (D/H)$$

d = upper-stem diameter inside bark (cm)

h = merchantable height (m)

D = outside bark dbh (cm)

H = total tree height (m)

$Z = h/H$

I = location of the inflection point

b_i = coefficient estimated using OLS and logarithmic transformation.

Model (39) can be expressed in linear form using a logarithmic transformation:

$$\begin{aligned}
\ln(d) = & \ln(b_0) + b_1 \ln(D) + \ln(b_2)D + b_3 \ln(X)Z^2 \\
& + b_4 \ln(X)\ln(Z+0.001) + b_5 \ln(X)\sqrt{Z} \\
& + b_6 \ln(X)e^Z + b_7 \ln(X)(D/H)
\end{aligned} \tag{40}$$

The seven independent variables in Kozak's model indicate that perhaps strong linear dependencies exist among them. In order to eliminate the multicollinearity problem, Perez et al. (1990) provided a reduced variable-form taper function for Pinus oocarpa based on Kozak's full model. The reduced model is expressed as:

$$d = b_0 D^{b_1} X^c \tag{41}$$

where

$$C = b_2 Z^2 + b_3 \ln(Z+0.001) + b_4 \ln(D/H).$$

Model (41) was linearized by making a logarithmic transformation:

$$\begin{aligned}
\ln d = & \ln(b_0) + b_1 \ln(D) + b_2 \ln(X)Z^2 \\
& + b_3 \ln(X)\ln(Z+0.001) + b_4 \ln(X)(D/H)
\end{aligned} \tag{42}$$

The Perez et al. (1990) variable-form taper function model (41) was selected as the taper function model for ponderosa pine and white fir in this study. Ponderosa pine data from 12 national forests and white fir data from 7 national forests

were fitted to model (42) using ordinary least squares regression. A slightly different reduced model from the one used by Perez et al. (1990) was fitted to Douglas-fir data. The linearized logarithmic taper function for Douglas-fir is expressed as:

$$\begin{aligned} \ln d = & \ln(b_0) + b_1 \ln(D) + b_2 \ln(X) Z^2 \\ & + b_3 \ln(X) \ln(Z+0.001) + b_5 \ln(X) e^Z \end{aligned} \quad (43)$$

The independent variables of equations (42) and (43) are the same as those defined for the Kozak (1988) taper function, with the exception that they are represented in English units. The inflection point for all three species is 25%. The estimated coefficients by species and national forest from OLS regression are listed in Tables 74-75.

Table 74
Estimated Coefficients of the Variable-form Taper
Function from OLS Regression by National Forest

Ponderosa Pine

National Forest	$\ln b_0$	b_1	b_2	b_3	b_4	Root MSE	R^2	Number of obs. (N)
Tonto	-0.5254	1.0955	0.4989	-0.0732	0.8430	0.1621	0.9571	849
A/S	-0.3626	1.0473	0.5949	-0.0546	0.7154	0.1704	0.9583	6898
Santa Fe	-0.2854	1.0320	0.7134	-0.0543	0.3230	0.2394	0.9197	2281
Prescott	-0.4646	1.0580	0.5960	-0.0803	0.7175	0.1492	0.9686	2219
Coconino	-0.3198	1.0231	0.3039	-0.0590	0.8776	0.1525	0.9629	2699
Kaibab	-0.3330	1.0321	0.4900	-0.0290	0.8807	0.1705	0.9553	2376
Lincoln	-0.1196	0.9686	0.5984	-0.0751	0.6632	0.1284	0.9721	2169
Dixie	-0.4284	1.0656	0.6842	-0.0714	0.1259	0.0715	0.9787	83
Carson	-0.3348	1.0272	0.6972	-0.0662	0.4859	0.1330	0.9711	1533
Rio Grande	-0.4903	1.1680	0.6800	-0.1140	0.5789	0.0628	0.9872	40
San Juan	-0.1901	0.9893	0.4451	-0.1015	0.7823	0.1467	0.9534	465
Gila	-0.1236	0.9498	0.4773	-0.0748	0.5323	0.1601	0.9090	215

Note: A/S = Apache-Sitgreaves National Forest

Taper model for inside bark upper-stem diameter:

$$\ln d = \ln b_0 + b_1 \ln D + b_2 \ln(X) Z^2 + b_3 \ln(X) \ln(Z+0.001) + b_4 \ln(X) (D/H)$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}) \quad Z = h/H \quad I = \text{inflection point}(0.25).$$

Table 75
Estimated Coefficients of the Variable-form Taper
Function from OLS Regression by National Forest

Douglas-fir

National Forest	$\ln b_0$	b_1	b_2	b_3	b_5	Root MSE	R^2	Number of obs. (N)
A/S	-0.2231	0.9617	0.3741	-0.0859	0.1750	0.1066	0.9828	325
Santa Fe	-0.1284	0.9286	0.6685	-0.1217	0.0376	0.1023	0.9646	816
Kaibab	0.2888	0.7983	0.3162	-0.0955	0.1595	0.1150	0.9711	350
Lincoln	-0.2363	0.9639	0.4582	-0.1105	0.1436	0.1463	0.9769	1844
Dixie	0.1625	0.8651	-0.3739	-0.0338	0.3695	0.0784	0.9719	91
Carson	0.2551	0.7975	0.4158	-0.0794	0.1397	0.1103	0.9753	175
Rio Grande	0.2790	0.8515	0.0764	-0.1092	0.2513	0.1206	0.9626	347

Note: A/S = Apache-Sitgreaves National Forest

Taper model for inside bark upper-stem diameter:

$$\ln d = \ln b_0 + b_1 \ln D + b_2 \ln(X) Z^2 + b_3 \ln(X) \ln(Z+0.001) + b_5 \ln(X) e^Z$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}); \quad Z = h/H; \quad \text{and } I = \text{inflection point } (0.25).$$

Table 76
Estimated Coefficients of the Variable-form Taper
Function from OLS Regression by National Forest

White fir

National Forest	$\ln b_0$	b_1	b_2	b_3	b_4	Root MSE	R^2	Number of obs. (N)
A/S	-0.1057	0.9259	0.5835	-0.0980	0.7918	0.1105	0.9775	264
Santa Fe	-0.2814	0.9913	0.4263	-0.0411	1.4353	0.1013	0.9789	148
Kaibab	0.0897	0.8663	0.6049	-0.1427	0.4693	0.1190	0.9657	285
Lincoln	-0.1680	0.9258	0.5639	-0.1190	0.8308	0.1175	0.9558	2554
Dixie	-0.2333	0.9888	0.5550	-0.0857	0.8327	0.1004	0.9570	99
Carson	0.2726	0.7983	0.4172	-0.0554	1.0052	0.1465	0.9545	122
Rio Grande	-0.9447	1.2822	0.4754	-0.1303	0.9786	0.1182	0.9658	32

Note: A/S = Apache-Sitgreaves National Forest

Taper model for inside bark upper-stem diameter:

$$\ln d = \ln b_0 + b_1 \ln D + b_2 \ln(X) Z^2 + b_3 \ln(X) \ln(Z+0.001) + b_4 \ln(X) (D/H)$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}); \quad Z = h/H; \quad \text{and } I = \text{inflection point } (0.25).$$

Modeling Hyperparameters The covariate selected for modeling the hyperparameters of ponderosa pine and white fir was the unit scaler, which implies that the EB estimate, $\tilde{B}_i$, shrinks toward a point. The model was as follows:

$$b_i' = Z \Gamma + \Delta_i \quad (44)$$

where

b_i' = a (1 x 5) vector of regression coefficients

from OLS estimation for national forest i

Z = an unit scaler

Γ = a (1 x 5) vector of coefficients of

hyperparameters

Δ_i = a (1 x 5) error vector.

The Douglas-fir hyperparameter model was given by:

$$b_i' = Z' \Gamma + \Delta_i \quad (45)$$

where

Z' = a (1 x 2) vector, $[1, Z_1]$

$$\begin{cases} Z_1 = 1 & \text{if sample tree is on the Dixie,} \\ & \text{or Rio Grande National Forest.} \\ Z_1 = 0 & \text{otherwise} \end{cases}$$

b_i' = a (1 x 5) vector of regression coefficients

for national forest i

Γ = a (2 x 5) matrix of coefficients of
hyperparameters

Δ_i = a (1 x 5) error vector.

Seemingly unrelated regression was used to estimate the hyperparameters. The estimated coefficients are listed in Tables 77-79.

EB Estimate The EB estimate for inside bark upper-stem diameter followed the procedure used previously in the total inside bark cubic foot volume estimation. The parameters of the EB estimate are listed in Tables 80-82, and the non-linear form of the variable-form taper functions are listed in Tables 83-85.

Table 77
Estimated Hyperparameter Coefficients of
Equation (44), and Associated t-test

Ponderosa Pine

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob > t
$\ln b_0$	Z	1	-0.3315	0.0390	-8.4890	0.0001
b_1	Z	1	1.0380	0.0167	62.1430	0.0001
b_2	Z	1	0.5649	0.0359	15.7320	0.0001
b_3	Z	1	-0.0711	0.0064	-11.1390	0.0001
b_4	Z	1	0.6271	0.0669	9.3810	0.0001

Note: Z = unit scaler

Table 78
Estimated Hyperparameter Coefficients of
Equation (45), and Associated t-tests

Douglas-fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
$\ln b_0$	intercept	1	0.0568	0.0916	0.620	0.5580
b_1	intercept	1	0.8809	0.0270	32.645	0.0001
b_2	intercept	1	0.4403	0.0735	5.990	0.0019
	Z_1	1	-0.5733	0.0726	-7.891	0.0005
b_3	intercept	1	-0.0909	0.0110	-8.240	0.0002
b_5	intercept	1	0.1341	0.0247	5.435	0.0029
	Z_1	1	0.1689	0.0302	5.591	0.0025

Note: The Γ (2 x 5) matrix may be expressed with the estimated coefficients as:

$$\Gamma = \begin{bmatrix} 0.0568 & 0.8809 & 0.4403 & -0.0909 & 0.1341 \\ 0 & 0 & -0.5733 & 0 & 0.1689 \end{bmatrix}$$

Table 79
Estimate Hyperparameter Coefficients of
Equation (44), and Associated t-tests

White fir

Dependent variable	Independent variable	Df	Parameter estimate (Γ)	Standard error	t for $H_0: \Gamma_i=0$	Prob $> t $
$\ln b_0$	Z	1	-0.1958	0.1447	-1.353	0.2247
b_1	Z	1	0.9684	0.0582	16.632	0.0001
b_2	Z	1	0.5180	0.0291	17.772	0.0001
b_3	Z	1	-0.0960	0.0144	-6.685	0.0005
b_4	Z	1	0.9063	0.1102	8.224	0.0002

Note: Z = unit scaler

Table 80
Parameters of the EB Estimate for the Variable-form
Taper Function by National Forest

Ponderosa Pine					
National Forest	$\ln \tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	$\tilde{B}_4$
Tonto	-0.4980	1.0862	0.5040	-0.0734	0.8265
A/S	-0.3622	1.0471	0.5960	-0.0554	0.7109
Santa Fe	-0.2935	1.0341	0.7128	-0.0571	0.3214
Prescott	-0.4598	1.0569	0.5931	-0.0776	0.7278
Coconino	-0.3181	1.0223	0.3071	-0.0606	0.8686
Kaibab	-0.3290	1.0300	0.4975	-0.0348	0.8546
Lincoln	-0.1315	0.9724	0.5979	-0.0749	0.6645
Dixie	-0.3961	1.0566	0.6831	-0.0685	0.1491
Carson	-0.3365	1.0279	0.6953	-0.0652	0.4926
Rio Grande	-0.4436	1.1477	0.6778	-0.1144	0.5644
San Juan	-0.2042	0.9939	0.4720	-0.1023	0.7229
Gila	-0.1887	0.9750	0.4649	-0.0690	0.5833

Note: A/S = Apache-Sitgreaves National Forest

Model for the EB estimate of inside bark upper-stem diameter:

$$\ln d = \ln \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln(X) Z^2 + \tilde{B}_3 \ln(X) \ln(Z+0.001) \\ + \tilde{B}_4 \ln(X) (D/H)$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}); \quad Z = h/H \\ I = \text{inflection point (0.25)}.$$

Table 81
Parameters of the EB Estimate for the Variable-form
Taper Function by National Forest

Douglas-fir

National Forest	$\ln \tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	$\tilde{B}_5$
A/S	-0.2168	0.9590	0.3907	-0.0897	0.1687
Santa Fe	-0.1196	0.9264	0.6500	-0.1188	0.0443
Kaibab	0.2778	0.8025	0.3269	-0.0919	0.1569
Lincoln	-0.2350	0.9635	0.4582	-0.1097	0.1437
Dixie	0.1630	0.8614	-0.2786	-0.0483	0.3381
Carson	0.2014	0.8165	0.3962	-0.0836	0.1453
Rio Grande	0.2779	0.8515	0.0636	-0.1073	0.2548

Note: A/S = Apache-Sitgreaves National Forest

Model for the EB estimate of inside bark upper-stem diameter:

$$\ln d = \ln \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln(X) Z^2 + \tilde{B}_3 \ln(X) \ln(Z+0.001) \\ + \tilde{B}_5 \ln(X) e^Z$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}); \quad Z = h/H \\ I = \text{inflection point (0.25)}.$$

Table 82
Parameters of the EB Estimate for the Variable-form
Taper Function by National Forest

White fir

National Forest	$\ln \tilde{B}_0$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	$\tilde{B}_4$
A/S	-0.1146	0.9287	0.5717	-0.0972	0.8236
Santa Fe	-0.2850	0.9903	0.4587	-0.0539	1.2935
Kaibab	0.0516	0.8792	0.6021	-0.1394	0.4742
Lincoln	-0.1618	0.9241	0.5637	-0.1181	0.8371
Dixie	-0.2096	0.9807	0.5037	-0.0824	0.9531
Carson	0.2698	0.7990	0.4505	-0.0666	0.9234
Rio Grande	-0.7926	1.2217	0.5103	-0.1335	0.8697

Note: A/S = Apache-Sitgreaves National Forest

Model for the EB estimate of inside bark upper-stem diameter:

$$\ln d = \ln \tilde{B}_0 + \tilde{B}_1 \ln D + \tilde{B}_2 \ln(X) Z^2 + \tilde{B}_3 \ln(X) \ln(Z+0.001) \\ + \tilde{B}_4 \ln(X) (D/H)$$

where

$$X = (1 - \sqrt{Z}) / (1 - \sqrt{I}); \quad Z = h/H \\ I = \text{inflection point (0.25)}.$$

Table 83
The Variable-form Inside Bark Taper Function
in Non-linear Form by National Forest

Ponderosa Pine

National Forest

Equation

Tonto

$$d = 0.6077D^{1.0862}X^{[0.5040Z^2 - 0.0734\ln(Z+0.001) + 0.8265(D/H)]}$$

Apache-Sitgreaves

$$d = 0.6961D^{1.0471}X^{[0.5960Z^2 - 0.0554\ln(Z+0.001) + 0.7109(D/H)]}$$

Santa Fe

$$d = 0.7456D^{1.0341}X^{[0.7128Z^2 - 0.0571\ln(Z+0.001) + 0.3214(D/H)]}$$

Prescott

$$d = 0.6314D^{1.0569}X^{[0.5931Z^2 - 0.0776\ln(Z+0.001) + 0.7278(D/H)]}$$

Coconino

$$d = 0.7275D^{1.0223}X^{[0.3071Z^2 - 0.0606\ln(Z+0.001) + 0.8686(D/H)]}$$

Kaibab

$$d = 0.7196D^{1.0300}X^{[0.4975Z^2 - 0.0348\ln(Z+0.001) + 0.8546(D/H)]}$$

Lincoln

$$d = 0.8768D^{0.9724}X^{[0.5979Z^2 - 0.0749\ln(Z+0.001) + 0.6645(D/H)]}$$

Table 83, continued

National Forest	Equation
Dixie	$d = 0.6729D^{1.0566}X^{[0.6831Z^2-0.0685\ln(Z+0.001)+0.1491(D/H)]}$
Carson	$d = 0.7143D^{1.0279}X^{[0.6953Z^2-0.0652\ln(Z+0.001)+0.4926(D/H)]}$
Rio Grande	$d = 0.6417D^{1.1477}X^{[0.6778Z^2-0.1144\ln(Z+0.001)+0.5644(D/H)]}$
San Juan	$d = 0.8153D^{0.9939}X^{[0.4720Z^2-0.1023\ln(Z+0.001)+0.7229(D/H)]}$
Gila	$d = 0.8280D^{0.9750}X^{[0.4649Z^2-0.0690\ln(Z+0.001)+0.5833(D/H)]}$

Note: $X = (1 - \sqrt{Z}) / (1 - \sqrt{I})$; $Z = h/H$
 $I = \text{inflection point } (0.25).$

Table 84
The Variable-form Inside Bark Taper Function
in Non-linear Form by National Forest

Douglas-fir

National Forest	Equation
-----------------	----------

Apache-Sitgreaves

$$d=0.8051D^{0.9590}X^{[0.3907Z^2-0.0897\ln(Z+0.001)+0.1687e^Z]}$$

Santa Fe

$$d=0.8873D^{0.9264}X^{[0.6500Z^2-0.1188\ln(Z+0.001)+0.0443e^Z]}$$

Kaibab

$$d=1.3202D^{0.8025}X^{[0.3269Z^2-0.0919\ln(Z+0.001)+0.1569e^Z]}$$

Lincoln

$$d=0.7906D^{0.9635}X^{[0.4582Z^2-0.1097\ln(Z+0.001)+0.1437e^Z]}$$

Dixie

$$d=1.1770D^{0.8614}X^{[-0.2786Z^2-0.0483\ln(Z+0.001)+0.3381e^Z]}$$

Table 84, continued

National Forest	Equation
Carson	$d=1.2231D^{0.8165}X^{[0.3962Z^2-0.0836\ln(Z+0.001)+0.1453e^Z]}$
Rio Grande	$d=1.3204D^{0.8515}X^{[0.0636Z^2-0.1073\ln(Z+0.001)+0.2548e^Z]}$

Note: $X = (1 - \sqrt{Z}) / (1 - \sqrt{I})$; $Z = h/H$
 $I = \text{inflection point } (0.25).$

Table 85
The Variable-form Inside Bark Taper Function
in Non-linear Form by National Forest

White fir

National Forest	Equation
-----------------	----------

Apache-Sitgreaves

$$d=0.8917D^{0.9287}X^{[0.5717Z^2-0.0972\ln(Z+0.001)+0.8236(D/H)]}$$

Santa Fe

$$d=0.7520D^{0.9903}X^{[0.4587Z^2-0.0539\ln(Z+0.001)+1.2935(D/H)]}$$

Kaibab

$$d=1.0530D^{0.8792}X^{[0.6021Z^2-0.1394\ln(Z+0.001)+0.4742(D/H)]}$$

Lincoln

$$d=0.8506D^{0.9241}X^{[0.5637Z^2-0.1181\ln(Z+0.001)+0.8371(D/H)]}$$

Dixie

$$d=0.8109D^{0.9807}X^{[0.5037Z^2-0.0824\ln(Z+0.001)+0.9531(D/H)]}$$

Table 85, continued

National Forest	Equation
Carson	$d=1.3097D^{0.7990}X^{[0.4505Z^2-0.0666\ln(Z+0.001)+0.9234(D/H)]}$
Rio Grande	$d=0.4527D^{1.2217}X^{[0.5103Z^2-0.1335\ln(Z+0.001)+0.8697(D/H)]}$

Note: $X = (1 - \sqrt{Z}) / (1 - \sqrt{I})$; $Z = h/H$
 $I = \text{inflection point } (0.25).$

Outside Bark Upper-stem
Diameter Equation

Voorhies et al. (1974) studied bark thickness of young-growth ponderosa pine from 27 stands in Northern Arizona and New Mexico. They developed an equation that predicts inside bark upper-stem diameter from outside bark upper-stem diameter. After inverting their equation, upper-stem, outside bark ponderosa pine diameter may be expressed as:

$$d_{ob} = 0.161 + 1.1933 d_{ib}$$

where

d_{ob} = outside bark upper-stem diameter in inches

d_{ib} = inside bark upper-stem diameter in inches.

It is apparent from the inspection of the residuals of the Voorhies et al. (1974) regression between d_{ib} and d_{ob} , that a weighted model would have been more appropriate. Therefore, the following model was used to predict outside bark upper-stem diameter as a function of inside bark upper-stem diameter:

$$d_{ob} = b_0 d_{ib}^{b_1} \quad (46)$$

The linearized model is expressed as:

$$\ln d_{ob} = \ln b_0 + b_1 \ln d_{ib} \quad (47)$$

The coefficients were estimated using least squares regression with a total of 1037 ponderosa pine observations from all national forests. These observations exclude the observed d_{ob} and d_{ib} measurements at stump height. The coefficient of determination for model (47) exceeded 0.995 for ponderosa pine. Equation (47) was also fitted to 2594 d_{ob} and d_{ib} observation pairs for Douglas-fir and 3089 observation pairs of white fir. The coefficient of determination exceeded 0.996 and 0.992 for Douglas-fir and

white fir respectively. The equation may be expressed in non-linear form with estimated coefficients as:

Ponderosa Pine

$$d_{ob} = 1.1890d_{ib}^{0.9765}$$

Douglas-fir

$$d_{ob} = 1.2063d_{ib}^{0.9751}$$

White fir

$$d_{ob} = 1.1654d_{ib}^{0.9962}$$

CHAPTER 5

CONCLUSION

Evaluation

We evaluated the predictive power of the volume models developed with each of three methods: (1) OLS for all national forests combined; (2) OLS for each individual national forest; and (3) Empirical Bayes on an individual national forest basis. Among the different species and various types of volume equations, the total inside bark cubic foot volume equation for ponderosa pine was selected for the evaluation procedure. The ponderosa pine data set was the largest in size and most extensive in geographical coverage. The total cubic foot inside bark volume equation was fitted with high precision, and the use of any other equation would involve the arbitrary selection of a merchantable limit. For testing purposes, ponderosa pine data from each national forest were randomly segregated into fitting data and testing data sets. Approximately 50 percent of the observations from each national forest were included in the fitting data set, and the remaining 50 percent were used as the testing data set. The total inside bark cubic foot volume model was fitted to the fitting data set by national forest using ordinary least squares (OLS) and the Empirical Bayes method. The model was also fitted to the data set with all national forests combined. Then, the fitted models were used to predict volume on the testing

data set. The bias and sums of squares of error (SSE) were selected as two statistics for comparing the predictive power of the three different methods, and were computed with the testing data set for each national forest. Bias is defined as the mean of observed minus predicted values in cubic feet, and the SSE is the sum of observed minus predicted values squared. The results of the bias statistic are presented in Table 86, the SSE are shown in Table 87. The overall average bias and SSE are listed in Table 88.

Conclusion

The values in Table 86 and Table 87 indicate that the EB models predicted about as well as the "OLS individual national forest" models on a forest-by forest basis when the sample size was relatively large, and much better than the combined OLS model regardless of sample size. However, the predictive power of the EB model was superior to both the OLS, on an individual national forest basis and by national forests combined, when the sample size was very small. The results in Table 88 show that on an overall basis, the EB models actually performed better than did the "OLS individual national forest" and than the "OLS national forests combined" when judged on the basis of the statistics of bias and SSE. The EB method yielded a 57 percent reduction in overall bias, and a 0.6 percent reduction in

overall SSE as opposed to the "OLS individual national forest" method.

Table 86
Bias from an Independent Test of the Total Inside Bark
Cubic Foot Volume Equation from OLS for all National
Forests Combined, OLS for Each Individual
Forest, and EB for Each National Forest

National Forest	OLS individual-NF		EB individual-NF		OLS NF-combined	
	N	Bias	N	Bias	N	Bias
Tonto	31	-0.0472	31	-0.0483	953	-0.0407
A/S	332	-0.0083	332	-0.0082	953	0.0035
Santa Fe	82	-0.0160	82	-0.0149	953	0.0429
Prescott	78	-0.0314	78	-0.0337	953	-0.1288
Coconino	128	0.0053	128	0.0045	953	-0.0094
Kaibab	83	-0.0009	83	-0.0012	953	-0.0285
Lincoln	74	-0.0121	74	-0.0114	953	0.0066
Dixie	11	0.0249	11	-0.0145	953	0.0300
Carson	56	-0.0165	56	-0.0190	953	-0.0717
Rio Grande	6	0.0232	6	0.0158	953	0.3972
San Juan	40	-0.0024	40	-0.0006	953	0.1179
Gila	22	0.0001	22	0.0023	953	0.0550

Note: A/S = Apache-Sitgreaves
N = Fitting sample size
NF = National Forest
Bias = mean of (observed - predicted) in
cubic feet.

Table 87
SSE from an Independent Test of the Total Inside Bark
Cubic Foot Volume Equation from OLS for all National
Forests Combined, OLS for Each Individual
Forest, and EB for Each National Forest

National Forest	OLS individual-NF		EB individual-NF		OLS NF-combined	
	N	SSE	N	SSE	N	SSE
Tonto	31	0.5026	31	0.5100	953	0.5308
A/S	332	8.0943	332	8.0998	953	8.2573
Santa Fe	82	0.8706	82	0.8710	953	1.0943
Prescott	78	1.2690	78	1.2575	953	2.5186
Coconino	128	1.5016	128	1.4942	953	1.5157
Kaibab	83	1.3436	83	1.3555	953	1.4292
Lincoln	74	0.7786	74	0.7683	953	0.7351
Dixie	11	0.2400	11	0.2247	953	0.2443
Carson	56	0.7853	56	0.7895	953	1.0607
Rio Grande	6	0.0329	6	0.0081	953	0.7912
San Juan	40	0.5695	40	0.5043	953	1.0604
Gila	22	0.2853	22	0.2973	953	0.4129

Note: A/S = Apache-Sitgreaves
 NF = National Forest
 N = Fitting sample size
 SSE = Sum of the (observed-predicted)².

Table 88
Overall Average Bias and SSE by
Estimation Procedure

Estimation procedure	Bias	SSE
OLS individual-NF	-0.0063	1.3562
EB individual-NF	-0.0027	1.3484
OLS NF-combined	0.0312	1.6375

The EB estimate approach described in this study has an advantage in the development of volume equations when the sample size is very small. The EB model provided a more precise estimate than OLS in this study, and the EB estimate results in unique, localized estimates while the combined national forest OLS estimate did not. It appears that the EB method will be useful in other situations where a number of vectors are to be estimated and the respective sample sizes are relatively small. We concluded that the EB estimates for ponderosa pine volume were superior to the OLS individual national forest models in this case. We further infer that the total and merchantable volume models and taper functions for southwestern ponderosa pine, Douglas-fir, and white fir developed with the EB method in this study are superior to those previously reported volume models developed with the ordinary least squares approach.

LITERATURE CITED

- Avery, T. E., and H. E. Burkhardt. 1983. Forest measurements. McGraw-Hill Book Company. New York. 46p.
- Bruce, D. 1982. Butt log volume estimators. For. Sci. 28:489-503.
- Burk, T. E., and A. R. Ek. 1982. Application of Empirical Bayes / James-Stein procedures to simultaneous estimation problems in forest inventory. Forest Sci. 28:753-771.
- Burkhardt, H. E. 1977. Cubic-foot volume of loblolly pine to an any merchantable top limit. Southern J. Appl. For. 1:7-9.
- Cao, Q. V., H. E. Burkhardt, and T. A. Max. 1980. Evaluation of two methods for cubic volume prediction of loblolly pine to any merchantable limit. Forest Sci. 26:71-80.
- Cao, Q. V., and H. E. Burkhardt. 1980. Cubic-foot volume of loblolly pine to any height limit. Southern J. Appl. For. 4:166-169.
- Chojnacky, D. C. 1988. Junipor, Pinyon. Oak, and Mesquite volume equations for Arizona. USDA For. Serv. Res. Paper INT-391.
- Clutter, J. L. 1980. Development of taper functions from variable top merchantable volume equations. Forest Sci. 26: 117-120.
- Dempster, A. P., N. M. Laird, and D. B. Rubin. 1977. Maximum likelihood from incomplete data via the EM algorithm (with discussion). J. Royal Stat. Soc. Ser. B:39:1-38.
- Ek, A. R., and J. N. Issos. 1978. Bayesian theory and multi-resource inventory. In Integrated inventories of renewable natural resources: Proceedings of the workshop, p.291-298. USDA For. Serv. Gen. Tech. Rep. RM-55, 482p.

- Fahey, T. D., T. A. Max, and J. K. A. Sachet. 1986. Factors affecting grade change of ponderosa lumber in the desert Southwest. *Forest Prod. J.* 36(6):36-40.
- Ffolliott, P.F., F.R. Larson, and R.L. Barger. 1971. Predicting scaled volume recoverable from cutover southwestern ponderosa pine stands. *USDA For. Serv. Res. Note RM-195.*
- Gallant, A. R., and W. A. Fuller. 1973. Fitting segmented polynomial models whose join points have to be estimated. *J. Am. Stat. Assoc.* 68:144-147.
- Green, E. J. and W. E. Strawderman. 1985. The use of Bayes/Empirical Bayes estimation in individual tree equation development. *Forest Sci.* 31(4): 975-990.
- Green, E. J., and W. E. Strawderman. 1986. Stein-rule estimation of coefficients for 18 eastern hardwood cubic volume equations. *Can. J. For. Res.* 16:249-255.
- Green, E. J., W. E. Strawderman, and C. E. Thomas. 1990. Empirical Bayes methods for calibrating yield models to multiple regions. Presented at the IUFRO Forest Simulation Systems Conference, Berkeley, California, November 2-5, 1988. *Univ. of Ca., Div. of Agric. and Nat. Res., Bull.* 1927 pp 89-96.
- Hann, D. W., and B. B. Bare. 1978. Comprehensive tree volume equations for major tree species of New Mexico and Arizona: I Results and Methodology. *USDA For. Serv. Res. Paper INT-209.*
- Hann, D. W. 1980. Development and evaluation of a even- and uneven-aged ponderosa pine/Arizona fescue stand simulator. *USDA For. Serv. Res. Paper INT-209.*
- Hornibrook, E. M. 1936. Scribner volume tables for cut-over stands of ponderosa pine in Arizona. *J. Agric. Res.* 52(12):961-974.
- Jameson, D. A. 1969. Rainfall patterns on vegetation zones in Northern Arizona. *Plateau* 41: 105-111.
- Kemp, P. D. 1957. Regression coefficients for computing cubic-foot volume of Rocky Mountain trees. *USDA For. Serv. Res. Paper.* INT-40.
- Knoebel, B. R. , H. E. Burkhardt, and D. E. Beck. 1984. Stem volume and taper functions for yellow-poplar in the Southern Appalachians. *Southern J. Appl. For.* 8:185-188.

- Krauch, H., and G. Peterson. 1943. Two new board-foot volume tables for Douglas-fir. USDA For. Serv. Southwestern For. and Range Exp. Stn. Res. Note 107
- Kozak, A. 1988. A variable-exponent taper equation. Can. J. For. Res. 18:1363-1368.
- Lexen, B.R., and W.G. Thomson. 1938. White fir merchantable height volume table. USDA For. Serv. Southwestern For. and Range Exp. Stn. Res. Note 27
- Max, T. A., and H. E. Burkhart. 1976. Segmented polynomial regression applied to taper equations. Forest Sci. 22:283-289.
- McTague, J. P., and W. F. Stansfield. 1988. Total and merchantable volume and taper functions for southwestern ponderosa pine. West. J. Appl. For. 3(4):123-125.
- Minor, C. O. 1961. Pulpwood volume tables for ponderosa pine in Arizona. USDA For. Serv. Res. Note 69, 6p. Rocky Mt. For. and Range Exp. Stn.. Fort Collins, Colo.
- Morris, C. 1983. Parametric empirical Bayes inference: Theory and applications (with discussion). J. Am. Stat. Assoc. 78:47-65.
- Myers, C. A. 1963. Volume, taper, and related tables for southwestern ponderosa pine. USDA For. Serv. Res. Paper RM-2 (revised in 1972).
- Newnham, R. M. 1988. A variable-form taper function. Petawawa Nat. For. Inst. Can. For. Serv. Info. Rep. PI-X-83. 33p.
- Ormerod, D. W. 1973. A simple bole model. For. Chron. 49:136-138.
- Perez, D. N., H. E. Burkhart, and C. T. Stiff. 1990. A variable-form taper function for Pinus oocarpa(Schiede) in Central Honduras. Forest Sci. 36:186-191.
- Peterson, G. 1958. Board-foot volumes of white fir to an 8-inch top. USDA For. Serv. Rocky Mountain For. and Range Exp. Stn. Res. Note 30.
- Staebler, G. R. 1952. The formula Scribner log rule. USDA For. Serv. Res. Notes PNW-78.
- USDA Forest Service. 1973. Timber sale preparation handbook. Southwestern Region. FSH 2409.24 R-3.

- USDA Forest Service. 1977. Timber appraisal handbook. Intermountain Region. R-4 FSH 4/77 AMEND 249
- USDA Forest Service. 1984. Timber sale preparation handbook. Southwestern Region. R3-2409.245.
- Voorhies, G., M.D. Fleming, and C.E. Thomas. 1974. Bark characteristics of young-growth southwestern ponderosa pine. Northern Arizona University. School of Forestry. Arizona Forestry Notes. No. 12
- Zellner, A. 1962. An efficient method of estimating seemingly unrelated regressions and tests for aggregating bias. Journal of the American Statistical Association. 57: 348-368.
- Zellner, A., and W. Vandaale. 1975. Bayes-Stein estimators for k-means regression and simultaneous equation models. In: Studies in Bayesian econometrics and statistics, page 627-653, S. Fienberg and A. Zellner, editors. North-Holland Publishing Company, Amsterdam.